Transverse horizontal spatial coherence of deep arrivals at megameter ranges

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Predictions of transverse horizontal spatial coherence from path integral theory are compared with measurements for two ranges between 2000 and 3000 km. The measurements derive from a low-frequency (75 Hz) bottom-mounted source at depth 810 m near Kauai that transmitted m-sequence signals over several years to two bottom-mounted horizontal line arrays in the North Pacific. In this paper we consider the early arriving portion of the deep acoustic field at these arrays. Horizontal coherence length estimates, on the order of 400 m, show good agreement with lengths calculated from theory. These lengths correspond to about 1° in horizontal arrival angle variability using a simple, extended, spatially incoherent source model. Estimates of scintillation index, log-amplitude variance, and decibel intensity variance indicate that the fields were partially saturated. There was no significant seasonal variability in these measures. The scintillation index predictions agree quite well with the dataset estimates; nevertheless, the scattering regime predictions (fully saturated) vary from the regime classification (partially saturated) inferred from observation. This contradictory result suggests that a fuller characterization of scattering regime metrics may be required. © 2005 Acoustical Society of America. [DOI: 10.1121/1.1854851]

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I. INTRODUCTION

The theory of wave propagation in random media (WPRM) has had considerable success in its original context involving fields propagating through homogeneous turbulent media.1–3 The application to long-range propagation in the ocean requires modifications: the medium is a waveguide, the background sound speed is depth dependent, and the sound speed fluctuations are inhomogeneous and anisotropic. Flatté et al.4 developed a theory (henceforth FDMWZ) for signal statistics that incorporated these additional features. Predictions based on FDMWZ theory have been compared to measurements from the straits of Florida,5 Cobb seamount,6 the Azores,7 and the Slice89 experiments. These experiments involved ranges up to 1000 km and used single hydrophones or multiple hydrophones attached to vertical moorings (i.e., vertical arrays). The Acoustic Thermometry of Ocean/Climate (ATOC) Engineering Test (AET) experiment6 tested the theory over a much longer range of 3250 km, but again only vertical line arrays were used to sample the acoustic field.

Second moment measurements available with these configurations were temporal (time–time) correlations, frequency–frequency correlations, and vertical spatial correlations. A measurement not available in these experiments due to equipment configurations was the horizontal second spatial moment, i.e., the horizontal coherence. An opportunity to measure horizontal coherence at 75 Hz over multi-megameter paths arose during the ATOC and follow-on North Pacific Acoustic Laboratory (NPAL) experiments.10 In this paper we report on the use of the data from two NPAL horizontal line array receivers at ranges from 2000 to 3000 km for measuring the transverse horizontal coherence.

The type of signals under consideration here are the long-range low-frequency ATOC signals trapped in the SOFAR channel. This is essentially a waveguide problem; as is characteristic of a waveguide problem, the acoustic field separates into faster modes with high-angle wavenumber vectors and slower modes with low-angle wavenumber vectors. The partial fields associated with the faster modes are well characterized by ray theory, and these manifest themselves as earlier, ray-like arrivals. Partial fields associated with the slower modes are more mode-like, and comprise the late crescendo11,12 (sometimes called the finale or coda). This ray-mode duality is nicely demonstrated by Grabb.13

The full acoustic field at the receiver can thus be characterized roughly as

\[ p = p_1 + p_2 + \cdots + p_{\text{modal}}, \]

where \( p_1 \) is the “first” ray-like arrival, \( p_2 \) is the next ray-like arrival, and so on. \( p_{\text{modal}} \) represents the final crescendo, and the explicit space–time dependence on location \( r \) and time \( t \) has been suppressed. While rays and modes have been utilized for tomography in both deep and shallow water, the early arrivals in this study are best modeled by rays.14 The statistical parameters—notably, in this paper, the spatial coherence—of wave fronts associated with individual ray-like arrivals are quantities of fundamental importance.

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teristics of the ocean by constructing the deterministic eigen-
ray from source to receiver and considering sound speed fluctua-
tions along that path to be the primary contribution to the
path integral calculation. (A “deterministic eigenray” is
conceptualized here as an eigenray traced through an ocean
devoid of sound speed fluctuations.) Inhomogeneous and an-
isotropic sound speed fluctuations are represented by the
Garrett–Munk interval wave spectrum.\textsuperscript{16–18} Fluctuation cal-
culations have been implemented numerically by the code
“Calculations of Acoustic Fluctuations due to Interval
Waves” (CAFI).\textsuperscript{19,20} Empirical measurements in this paper
are compared to FDMWZ predictions calculated by CAFI.

The remainder of the paper is organized as follows. In
Sec. II we present the definitions of the mathematical enti-
ties that will represent the coherence and the intensity measures
predicted by theory. In Sec. III we describe the NPAL experi-
ment, including the \textit{m}-sequence signal parameters. The data
processing is described in Sec. IV: this concerns the calcula-
tion of quantities that are comparable to theoretical quanti-
ties. Experimental results are shown in Sec. V for the stron-
gest deep arrival over each of two long-range paths. CAFI
calculations for fluctuation statistics over representative
paths of similar length are given in Sec. VI, with an analysis
of the findings in Sec. VII. The findings indicate that the
horizontal coherence length and scintillation index estimates
are found to agree with CAFI predictions. Scattering regime
classifications do not agree, however: the regime is inferred
from actual measurements to be partially saturated, but CAFI
identifies the regime as fully saturated. This contradictory
result suggests that a fuller characterization of scattering re-
gime metrics may be required. The main results are summa-
rized in Sec. VIII.

II. THEORY

A. Coherence

In general, coherence is a measure of correlation, with
values near one indicating a high correlation and values near
zero poor correlation. However, there are subtle variations in
the interpretation of coherence among investigators from dif-
ferent specialities. Motivated by these variations, the follow-
ing discussion reviews several notions of coherence as a
background for the definitions used in this analysis.

Coherence is generally a normalized second moment. Letting \( \psi(r, t) \) represent a general complex scalar field
(e.g., an acoustic field) at location \( r \) and time \( t \), and using the shorthand notation \( \psi(l) = \psi(r_l, t_l) \), the second moment is

\[
M_{\phi\phi}(l, m) = \langle \psi^*(l) \psi(m) \rangle,
\]

where \( \langle \cdot \rangle^* \) denotes conjugation and \( \langle \cdot \rangle \) expectation. This is the complex cross-correlation from elementary statistics. In
the statistical optics literature,\textsuperscript{21,22} the degree of complex co-
herence is defined as Eq. (2), normalized by the “self-” (or autocorrelation) terms:

\[
\gamma_{lm} = \frac{M_{\phi\phi}(l, m)}{\sqrt{M_{\phi\phi}(l, l)M_{\phi\phi}(m, m)}}.
\]

This is essentially a complex correlation coefficient with a
magnitude in the range \([0,1]\). \( |\gamma_{lm}| = 1 \) implies full coher-

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Strength of time front arrivals at site N from a source mounted on a
Pioneer seamount. (Site designations are shown later in this paper.) Top:
time fronts at the range of site N. The receiver depth is suggested by the
horizontal line. Bottom: the signal to noise ratio (SNR) versus arrival time
scatterplots for 15 months of data. A correspondence between time front
arrivals and strong SNR arrivals is indicated by dotted lines. The strongest
SNR arrivals appear to be related to time fronts with cusps very near the
receiver depth.}
\end{figure}
ence, and $|\gamma_{lm}| = 0$ implies complete incoherence.

In the theoretical literature of wave propagation in random media (WPRM), it is common to assume that the fields are normalized (i.e., the self-terms are 1), and

$$\gamma_{lm} = \Gamma(r_l, r_m, t_l, t_m),$$

(4)
is usually referred to simply as the mutual coherence. This could, in general, be complex, but often only its magnitude is of interest: in this paper, it is necessary to consider the full complex coherence. Note also that this quantity can involve any two stochastic processes, $\psi(r_l, t_l)$ and $\psi(r_m, t_m)$, and is a time–domain definition.

A quantity similar to Eq. (3) appears in time series analysis, where it is simply called the coherence. Here, the moments are cross-spectra ($G_{xy}(f)$) and autospectra ($G_{xx}(f)$ and $G_{yy}(f)$) between two time series $x(t)$ and $y(t)$:

$$\gamma_{xy}(f) = \frac{G_{xy}(f)}{\sqrt{G_{xx}(f)G_{yy}(f)}}.$$  

By comparison with Eq. (3), this is clearly seen to be a frequency domain version of the degree of complex coherence. One often sees the magnitude squared coherence $|\gamma_{xy}(f)|^2$ (MSC) in the literature.

Again, a similar quantity appears in statistical array processing. The elements of the array cross-spectral density matrix are precisely the second moments between sensors $l$ and $m$, with $\psi$ representing the space–time acoustic field. The moment is again considered to be a narrow band or single-frequency concept. The complexity here must be retained: beamforming depends on it. Cox has shown that the loss of signal coherence across an array is a fundamental limiting factor in array system performance.

The fundamental link between spatial coherence and array system performance motivates this analysis to adopt the frequency–domain, “single-frequency” interpretation of coherence. This choice enables a simpler and more direct mathematical discussion in this section, and also facilitates a direct comparison with a theoretical prediction in Sec. VI. There is unfortunately one main disadvantage: the data involve wideband signals and are commonly considered in the time–domain. (Indeed, after pulse compression, these kinds of experiments are usually considered pulse-propagation experiments.) The procedures and consequences for evaluating time–domain observations in a frequency–domain context are discussed in Sec. IV.

Following FDMWZ, let the acoustic pressure field at frequency $\omega$ be written as

$$p(r, t; \omega) = p_0(r_0, t_0; \omega)e^{i\mathbf{k} \cdot (r - r_0) - \omega(t - t_0)}\psi(r, t; \omega).$$

(5)

Here, $p_0$ is the deterministic part of the propagation to some reference position $r_0$ at time $t_0$. $\mathbf{k}$ is a “mean” incident wave number vector and $\omega$ is a “mean” frequency. The exponential term provides a phase correction to $p_0$ to yield the deterministic field at $r$ at time $t$. All the fluctuations are represented by $\psi$; this factor converts the deterministic field at $r$ and $t$ to a stochastic field $p(r, t)$. $\psi$ can contain randomness in amplitude and phase, including perturbations to $\mathbf{k}$ (due, e.g., to refracting media or extended sources) and $\omega$ (due to Doppler effects). The exponential term is required for beamforming and coherent summing, so we retain it in its full form where necessary to show how it affects various intermediate quantities. By definition, $\psi$ is normalized such that $\langle |\psi|^2 \rangle = 1$. The second moment expression [Eq. (2)] now becomes (dropping the $\omega$ notation)

$$M_{pp}(l, m) = \langle p^*_{x}(r_l, t_l) p(r_m, t_m) \rangle$$

$$= |p_0(r_0, t_0)|^2 e^{i\mathbf{k} \cdot (r_m - r_l)}$$

$$\times \langle \psi_{x}^*(r_l, t_l) \psi(r_m, t_m) \rangle.$$  

(6)

In this and in the following development, the fields are considered only at the same time, i.e., both at $t = t_0$, and the time notation is dropped. The mutual coherence is therefore

$$\Gamma_{lm} = \frac{\langle p_{x}^* p_m \rangle}{\sqrt{\langle |p_{x}|^2 \rangle \langle |p_m|^2 \rangle}}$$

$$= e^{i\mathbf{k} \cdot (r_m - r_l)} \langle \psi_{x}^* \psi_m \rangle$$

$$= e^{i\mathbf{k} \cdot (r_m - r_l)} \langle \psi_{x} \psi_m \rangle,$$  

(7)

where the last line follows because $\langle |\psi|^2 \rangle = \langle |\psi_n|^2 \rangle = 1$ by definition. When the phase fluctuations of the product $\psi_{x} \psi_m$ are zero mean, the expectation in the final expression above will be real. In the more general case, the expectation may provide a residual phase and hence be complex. Residual phase and the deterministic phase correction are eliminated by considering in this paper only the mutual coherence magnitude. This expression is formally valid at the single frequency $\omega$, although, in practice, there will be some finite bandwidth associated with actual measurements.

Some assumptions can be invoked to simplify the problem. The mutual coherence is generally a six-dimensional function of spatial coordinates. When the field $\psi$ is homogeneous, the mutual coherence is only a function of the difference $r_l - r_m$:

$$\Gamma(r_l, r_m) = \Gamma(R).$$

(8)

In general, $\psi$ is definitely not homogeneous everywhere in a waveguide. The sound speed fluctuations themselves are not homogeneous: in the vertical, for example, they are largest near the surface and decrease rapidly with depth. However, since the horizontal scales of the sound speed fluctuations are orders of magnitude larger than the vertical scales, it is therefore reasonable to assume that homogeneity holds, at least approximately, for horizontal sensor configurations of modest size compared to horizontal sound speed fluctuation correlation lengths, as is the case here.

While the self-term $\Gamma(R,R)$ = 1 by definition, the incoherent limit may not be 0. Conceptually, at great separation, (i.e., where $R = |R| \to \infty$),

$$\lim_{R \to \infty} \langle \psi_{x}^*(r_l, t) \psi_{x}^*(r_m, t) \rangle = \langle \psi_{x}^*(r_l, t) \rangle \langle \psi_{x}^*(r_m, t) \rangle$$

$$= \langle |\psi_{x}(r)|^2 \rangle = \Gamma_{x}.$$  

(9)
The term $\Gamma_{\infty}$ is real. Thus, the coherence at great separation should reach an asymptotic value proportional to the squared magnitude of the mean field (excluding attenuation). In the saturated regime, the mean field is identically zero, and it will turn out that, for these measurements, the mean field estimate is insignificantly different from zero. But it is worthwhile to note that it is incorrect to assume a priori that the mean field is zero, especially if the fluctuation regime is partially saturated or unsaturated.

Inasmuch as the receivers used here are nearly perfect horizontal line arrays, $\mathbf{R}$ will be modeled as some $R\hat{a}$, where $\hat{a}$ is a unit vector aligned along the mean bearing of the array. Thus

$$\Gamma(\mathbf{R}) = \Gamma(R\hat{a}).$$

Furthermore, for all horizontal line arrays, $\hat{a} \perp \hat{z}$ (where $\hat{z}$ is the vertical unit vector.) For transverse measurements of coherence, $\hat{a}$ must also be perpendicular to $\hat{k}$. This is not the case for the two arrays used here, so a transverse separation $u$ is defined that is related to the actual separation along the array via the projection

$$u = R\hat{a}\hat{t},$$

where $\hat{t}$ is a unit vector transverse to the propagation such that $\hat{t} \perp \hat{k}$ and $\hat{t} \perp \hat{z}$.

Combining Eqs. (8), (9), (10), and (11), a general expression for the transverse mutual coherence magnitude can be written as

$$|\Gamma(u)| = (1 - \Gamma_{\infty})g(u) + \Gamma_{\infty}.$$  

This expression involves a real, unknown "shape" function $g(u)$ with the properties

$$g(u) = \begin{cases} 
1, & u = 0, \\
0, & u \rightarrow \infty.
\end{cases}$$

The approach taken here is to model the unknown shape functions with the family $\hat{g}(u; \alpha)$, based on the single parameter $\alpha$. A model $\hat{g}$ is fit in the least-square sense to the data, yielding best-fit parameter $\alpha_{\text{LSE}} = \alpha_D$, and this function is shown for comparison against the spatial coherence function prediction produced by CAFI. A "transverse horizontal coherence length" $L_h$ is defined such that

$$\hat{g}(L_h; \alpha_{\text{LSE}} = \alpha_D) = e^{-1/2}.$$  

Another common definition of coherence length would be the "e-folding" length, the value at which the coherence is $e^{-1}$, but the definition in Eq. (14) will prove more convenient in Sec. VI. The estimate $\hat{L}_h$ using $\alpha_D$ will be compared to $L_h$ predicted by CAFI.

B. Scintillation index

The scintillation index $SI$ is defined as

$$SI = \text{var}(I)/\langle I \rangle^2 = \frac{\langle I^2 \rangle - \langle I \rangle^2}{\langle I \rangle^2} = \frac{\langle I^2 \rangle}{\langle I \rangle^2} - 1,$$

and characterizes the variability of the acoustic intensity $I$. Although apparently a simple formula, it is important to recognize that the variance cannot be simply computed from the available time series if the measurements are correlated in time. Moreover, the transmissions were spaced irregularly in time, and this leads to additional interpretational complexities described below.

III. EXPERIMENT

The locations of the instruments germane to the analysis presented in this paper are shown in Fig. 2. The Kauai source and receivers N and O were components of the larger NPAL experiment. The Pioneer source is shown for reference, but no further results for Pioneer transmissions beyond Fig. 1 are presented here.

The acoustic source located at Kauai was one of two very-low-frequency sources used in the ATOC/NPAL experiments. It was deployed in July 1997 on the northeast slope of Kauai at latitude 22°20.949′N and longitude 159°34.195′W at 810 m depth, and is cabled to shore at the Pacific Missile Range Facility. Built by Alliant Techsystems (Mukilteo, Washington), it is a pressure-compensated bender-bar/pipe-stave transducer designed for operation at depths to 1300 m. At 810 m depth, it has a main resonant frequency of 65 Hz and a $-3$ dB bandwidth of 14 Hz.

The source transmitted from August 1997 to October 1999, when the permit for its operation ended. Transmissions were nominally scheduled for every 4 h every fourth day, but this scheme varied widely. During the first one-third of the experiment, some transmissions were only two hours apart. There were several system outages that precluded data acquisition for more than a month at either receiver site (e.g., June 1998). A new permit was established in 2001, and the source resumed transmissions starting 1 January 2002 for a 5 year period. The data analyzed here correspond to transmissions acquired from October 1997 to the end of permitted operation in October 1999.

The transmissions were standard ATOC $m$-sequence signals. The signal parameters are provided in Table I. Data acquisition was timed to begin after the first $m$-sequence and conclude prior to the last $m$-sequence, thereby ensuring that the acoustic fields at the receivers were fully involved during...
data collection. The “principle transmission” consisted of 40 end-to-end sequences lasting 1091.20 s. The temporal and frequency characteristics are shown in Fig. 3.

Receivers N and O are bottom-mounted horizontal line arrays. The receivers are cabled to shore to equipment designed, built, and operated by the Applied Physics Laboratory, University of Washington (APL/UW). The signals were filtered between 2 Hz and 113.636 Hz (two-stage cascaded 6-pole Butterworth switched-capacitor filter), digitized at 300 Hz and spooled to disk. To reduce data storage requirements for each reception, every four consecutive m-sequences were coherently added, producing 10 “4-period sums” from each 40 m-sequence collection. The raw data was then discarded. All further data processing was conducted at APL/UW.

A total of 1204 receptions were collected at site N, and 1197 at site O. The temporal distributions of receptions at both sites are shown in Fig. 4.

IV. DATA PROCESSING

A. Preliminary processing

The first processing step (which occurred on the receiver computers) for each reception accumulated every four m-sequences into 4-period sums. For perfectly coherent signals in uncorrelated additive noise, this results in a processing gain of $10 \log_{10}(4) \approx 4$ dB. Since, from Table I, a single m-sequence has a duration of 27.28 s, this step represents a time average of the fluctuations over 109.12 s. Coherent summing enhances correlated features and deemphasizes short-duration changes, and therefore this step may result in a bias toward overestimating the coherence.

In the second step, the 4-period sums were pulse-compressed to achieve an additional theoretical gain of $10 \log_{10}(1023) \approx 30$ dB.

Since the quantity $\psi$ is usually defined in terms of solutions “at a single frequency” of the Helmholtz equation, the next step in processing was to develop “narrow band” quantities to represent $\psi$. Conceptually, after pulse compression, the arrival structure that corresponded to the partial field $p_j(r,t)$ would be isolated with a “time gate” and transformed into the Fourier domain with a discrete Fourier transform (DFT). $\psi$ is then represented by the single Fourier coefficient at the carrier frequency. The ideal time gate would contain all of the pulse arrival structure for the ray-like arrival under investigation, but not contributions from adjacent arrivals.

The spreading of each pulse and the proximity of pulse pairs to one another therefore rendered infeasible the construction of a time gate small enough to isolate one arrival from the other in a pair. It was, however, usually possible to

![FIG. 3. Features of m-sequence processing. Top: a section of a simulated m-sequence carrier at 75 Hz. (A complete m-sequence is 27.28 s in length.) Middle: envelope of the Fourier transform magnitude. Bottom: output magnitude after pulse compression (designed with an arrival time of 1.0 s).](image)

![FIG. 4. Receptions per month, Kauai to sites N (white) and O (black). 1204 total receptions at site N, 1197 at site O.](image)
isolate one pair from another. A time gate width of 0.5 s was adequate for this task, but this gate width unavoidably com- mingled the fluctuations from both pulses. This may result in a bias toward underestimating the coherence (of a single partial field).

The effective bandwidth of the Fourier coefficient is the reciprocal of the time gate width and is therefore 2 Hz. This is probably a fair representation of a single-frequency theoretical quantity, particularly as the coherent bandwidth of the channel is thought to be substantially larger.

Over the entire experiment, the data displayed considerable wander. Wander can be defined on several scales for these paths. Internal-wave-induced wander is roughly on the order of 10 ms; mesoscale-induced wander is roughly on the order of 100 ms; and gyrescale wander can produce wander on scales of 1000 ms.

If the small-scale wander produces a delay of \( \tau \) across all channels, this will introduce an additional factor of \( e^{-i\omega \tau} \) in Eq. (5). This factor, however, cancels in the second moment expression if the delay does not cause the pulse arrival structure to move out of the time gate.

The large-scale wander can move the pulse spread realization out of a fixed time gate. To automate processing, either the time gate must follow the large-scale wander, or the data must be synthetically shifted to correct for the large-scale wander. The gross behavior of the large-scale wander is called the track. All the data used here have been manually tracked, and the track solution used as an input to shift the pulse structure to the center of the time gate.

An obliquely incident acoustic field will introduce a factor of \( e^{i(k \cdot (r_j - r_0))} \) into the Fourier coefficient for channel \( j \). Here \( k \) is the wave number vector of the incident field. As long as the entire arrival structure falls within the same time gate for all sensor channels, and if \( k \parallel k \) is deterministic, Eq. (7) shows that \( |\Gamma| \) will be independent of obliquity.

In practice, for obliquely incident fields, a narrow time gate may exclude arrivals at the far ends of the array. The data in each channel have therefore been delayed (in the time domain) by a factor \( k_0 \cdot (r_j - r_0)/\omega_0 \), where \( k_0 \parallel \) is in the geo-desic direction from Kauai to the point \( r_0 \) on a WGS84 spheroid (This makes the signals appear as if they were incident from broadside).

(N.B.: If the incident wavenumber vector \( k \) is varying with time, due, say, to time-varying horizontal refraction, the data contain arrival angle wander. Unlike time domain wander, this influence does not factor out in calculations of \( |\Gamma| \). Averaging (i.e., expectation) in the presence of arrival angle wander will result in additional spatial decoherence.)

In summary, the processing steps from raw time domain data to \( \psi \) surrogate involved (1) four-period sums, (2) pulse compression, (3) large-scale wander correction, (4) obliquity correction, (5) time gate isolation, (6) Fourier transformation, and (7) coefficient extraction.

B. Coherence

It is possible to infer spatial coherence indirectly by examining the beamformed outputs of subarrays. However, receivers N and O measure individual sensor signals that are strong enough, after four-period summing and pulse-compression, to stand out from the background noise. This makes it possible to estimate the spatial coherence directly by cross-correlating \( \psi \) from pairs of individual sensors, which is the approach taken here.

The processing steps described in Sec. IV A yielded, for an array with \( N \) sensors, an \( N \times N \) vector \( \mathbf{v} \) of complex Fourier coefficients for the four-period sum group \( j \). This procedure was then repeated to obtain a vector \( \mathbf{v}_{j+1} \) for the four-period sum group \( j+1 \). These vectors \( \mathbf{v}_j \) and \( \mathbf{v}_{j+1} \) were then incorporated in an algorithm presented in Appendix A to form a coherence matrix estimate \( \mathbf{G} \) with elements

\[
[\mathbf{G}]_{jk} = |\Gamma(r_j - r_k)|.
\]

In FDWMZ theory, the transverse horizontal coherence magnitude is modeled as

\[
|\langle \psi^*(x) \psi(x + \Delta x) \rangle| = \exp \left( -\frac{1}{2} D(\Delta x) \right),
\]

where \( D(\Delta x) \) is a structure function (usually identified purely as a “phase structure function”) and \( \Delta x \) is the transverse horizontal spatial separation. For small separations, the structure function can be approximated by

\[
D(\Delta x) = (\Delta x/L_h)^{3/2},
\]

where \( L_h \) is the transverse horizontal coherence length. Small separations are those for which

\[
\Delta x \ll X_h \omega_l/\omega_X,
\]

where \( X_h \) is the typical horizontal correlation size of sound speed inhomogeneities, \( \omega_l \) is the local inertial frequency, and \( \omega_X^2 = \omega_s^2 + n^2 \tan^2 \beta \) depends on the local buoyancy frequency \( n \) and the inclination \( \beta \) of the local eigenray. Flatté and Stoughton suggest \( X_h \approx 12 \) km, and, using typical Northern Pacific ocean values for the other parameters, \( X_h \omega_l/\omega_X \approx 3.4 \) km. The sensor separations used in this analysis are generally less than this, and hence Eq. (18) is a valid approximation for the structure function here. Based on Eqs. (17) and (18), the model shape function \( \hat{g}(\Delta x) \) [cf., Eq. (13)] is

\[
\hat{g}(\Delta x; \alpha) = \exp \left( -\frac{1}{2} \left( \frac{\Delta x \hat{a}}{\alpha} \right)^{3/2} \right);
\]

Eq. (19) is combined with Eq. (12) to model the magnitude mutual coherence, and this is compared to Eq. (16). The model is fit to the data by minimizing over \( \alpha \) the sum of the squared errors between the model and the estimated coherence matrix. This produces the best-fit parameter \( \alpha_{\text{LSE}} = \alpha_D \), and the transverse spatial coherence length estimate \( L_h \) is then the root of

\[
(1 - \Gamma_{ee}) \hat{g}(\hat{L}_h; \alpha_D) + \Gamma_{ee} - e^{-1/2} = 0.
\]

C. Scintillation

Data processing for the scintillation index computation was slightly different. Additive ambient noise will increase the apparent variability in the data, and this will bias the scintillation index high. This bias can be reduced by beamforming, but beamforming will reduce the variability over that of a point measurement due to the averaging inherent in
the beamforming process. The advantage gained by beamforming generally outweighs the disadvantages, as long as the signal component across channels has a reasonably high cross-coherence. Therefore, each four-period sum was pulse compressed, corrected for large-scale wander, and delay-and-sum beamformed. The beamformer was steered to the geodesic path bearing from receiver to source assuming a WGS84 spheroid. The beamformer used only those channels with cross-coherence greater than 0.5. The appropriate pulse arrival structure was extracted and Fourier transformed. The amplitude of \( c \) was then represented by the modulus of the carrier Fourier coefficient and the intensity by the modulus squared. The log-amplitude and log-intensity were then computed accordingly.

The data sample therefore consists of the amplitude (intensity, log-amplitude, log-intensity) of the carrier Fourier coefficient for each four-period sum, beamformed to increase the signal-to-noise ratio. The mean intensity and variances are determined from this set. Ensemble averages are interpreted as arithmetic means over the sample population.

V. RESULTS

A. Site N

1. Data description

The ray-like arrival analyzed for site N had, on average, the best signal-to-noise ratio (SNR) of all the ray-like arrivals for this site. An estimated track for this arrival structure is shown in Fig. 5. The arrival time variation of roughly 0.5 s over the dataset can be attributed to meso- and gyrescale ocean processes.

The data at site N had generally poorer SNR than the data at site O, and therefore a further description of the appearance of the raw data is deferred to Sec. V B.

2. Coherence length

The asymptotic value \( \Gamma_\infty \) was estimated [from Eqs. (7) and (9)] using

\[
\Gamma_\infty = \frac{\langle |P| \rangle^2}{\langle |P|^2 \rangle}.
\]  

Figure 6: Histograms of bootstrap replicates of the asymptotic value \( \Gamma_\infty \). Left: Kauai to site N. The estimated value of 0.13 is not significantly different from 0.0. Right: Kauai to site O. The estimated value of 0.042 is not significantly different from 0.0.

For this calculation, the second four-period sum group in each reception was used. Motivated by the same argument advanced in Sec. IV C, the multichannel four-period sum data was beamformed to improve SNR. The bearing chosen was the bearing of the geodesic path to Kauai. The beamformed data was pulse compressed, time gated, and Fourier transformed. The estimated asymptotic value using the full two-year dataset was 0.13. The samples were bootstrapped, using 500 replicates, to approximate the distribution of this estimate. The histogram of the bootstrap replicates is shown in Fig. 6. Here 90% of the replicates were greater than the full sample estimate, so the asymptotic value \( |\Gamma_\infty| \) was judged to be not significantly different from zero.

The data covariance matrix was estimated following a procedure described in the Appendix that utilized four-period sum groups, two and three from each reception. The sample mutual coherence matrix was fit to Eq. (12) with \( \Gamma_\infty = 0 \). The coherence length estimate was 528 m. The error in this estimate, represented by the bootstrap standard error using 200 bootstrap replicates, was \( \pm 152 \) m.

3. Scintillation index

The scintillation index requires the variance of the intensity fluctuations, but care must be used in estimating the variance of a correlated sequence. The existence of a sample-to-sample correlation will generally bias the standard calculation. One alternative is to estimate the degrees of freedom in the signal and apply a corresponding correction. This would be a difficult undertaking because the raw four-period sums represent irregular clustered sampling in time. The
shortest sampling interval is from one four-period sum to the next within a transmission, or about 109 s. The clusters are generally spaced four hours apart on transmission days, sometimes less. Estimating correlation measures for this kind of irregular clustered data is nontrivial.

An alternative approach was used here. A new sample population was derived from the original by using only a single four-period sum in each reception.

The results used the second four-period sum. This time series is shown in Fig. 7. The nonparametric run test can be used to accept or reject the hypothesis that this new sample is random. The results are shown in Table II. When the new sample used all the available receptions, randomness was rejected at the 5% significance level. During the first third of the experiment, there were clusters of receptions only two hours apart. When receptions closer than four hours to the nearest neighbor were eliminated, the corresponding run test found the modified sample population to be random at the 5% level. Interestingly, this implies that the intensity was partially correlated on time scales of two hours.

Intensity fluctuation statistics (scintillation index, log-intensity variance, and decibel intensity variance) are shown in Table III for the modified sample population. In all cases, the quoted error is the bootstrap standard error. The variance of the decibel intensity was not significantly different than the Dyer value of 5.6 dB, but the scintillation index was significantly greater than the saturation value of 1.0. The log-intensity variance was approximately 2.0. The sample population was also partitioned by season to determine if there were significant seasonal effects, but the sample sizes were too small to yield statistically significant differences.

Based on the intensity statistics, signal fluctuations at site N were much stronger than would be expected for the Rytov regime. On the other hand, the scintillation index, which was greater than 1.0, shows that full saturation has not been achieved, even though the decibel-intensity variance was approximately the Dyer value. The regime was therefore classified as neither unsaturated nor saturated, but partially saturated.

B. Site O

1. Data description

The ray-like arrival analyzed for site O had, on average, the best SNR of all the ray-like arrivals at this site. An esti-
mated track for this arrival structure is shown in Fig. 8. Again, the arrival time variability of roughly 0.5 s over the dataset can be attributed to meso- and gyrescale ocean processes.

Representative raw (individual channel) pulse-compressed four-period summed arrivals are shown in Fig. 9 over a single reception (ten, four-period sums). A slow intensity change can be observed across the array over the ten pulses. The lower channels (say 1 to 5) had consistently higher signal power levels than the remaining channels. Since this was observed in data collected throughout the experiment, this was suspected to be due to gain mismatch. That this will not bias the results can be seen as follows. Let the generalized gain be represented by a complex channel-dependent coefficient $h_l$ for channel $l$. Then from Eq. (7), $G_{lm}$ would contain a term, $h_l^* h_m / \sqrt{|h_l|^2 |h_m|^2}$.

The magnitude of this term is 1, and therefore generalized gain mismatch has no influence on $|\Gamma_{lm}|$.

Representative pulse arrival structures are shown in Fig. 10. These figures show the variations in the individual realizations, ranging from a single clearly identifiable peak (where there should be a pair) to a cluster of low-level peaks where none are uniquely identifiable.

2. Coherence length

The asymptotic value $\Gamma_\infty$ was estimated using the second four-period sum group in each reception. The full population estimate was 0.042. A histogram of the bootstrap replicates is shown in Fig. 6. Here 98% of the replicates were greater than the full population estimate, so this value was also not significantly different from zero.

The data covariance matrix was estimated using four-period sum groups two and three for each transmission. The sample mutual coherence matrix was fit to Eq. (12) with $\Gamma_\infty = 0$. The coherence length estimate was 410 m, and the bootstrap error in this estimate using 200 replicates was $\pm 25$ m.

It is interesting to note that the coherence estimate from site N, which generally had data with a poorer SNR than that from site O, has a much greater variability, as evidenced by the bootstrap error. This is consistent with the convergence properties of the estimation algorithm in the Appendix: even though the estimator is unbiased, convergence still depends on SNR.

3. Scintillation index

The full sample population, derived from the original by using only the second four-period sum in each reception, is shown in Fig. 11. The nonparametric run test results are shown in Table IV. When the new sample population used all the available receptions, randomness was rejected at the 5% significance level. When receptions closer than 4 h to the nearest neighbor were eliminated, the corresponding run test found the modified sample population to be random at the 5% level.

Intensity fluctuation statistics are shown in Table V for the entire sample population. The quoted error is again the bootstrap standard error. The decibel intensity variance was again not significantly different than the Dyer value, but the scintillation index was significantly greater than the saturation value of 1.0. The log-intensity variance was again approximately 2.0. Again, there were no significant seasonal trends.

Based on the intensity statistics, signal fluctuations at site O were similar to site N and much stronger than those in the Rytov regime. Full saturation had not been achieved at
FIG. 11. Sample intensities, Kauai to site O. Top: intensity $I$ (normalized by the arithmetic mean intensity). Bottom: log intensity. The panels are partitioned by season, and annotated “W” for Winter, “V” for Spring, “S” for Summer, and “F” for Fall. The autumnal equinox in 1997 was September 22.

TABLE IV. Run test for randomness of the intensity fluctuations at site O. The randomness hypothesis is accepted at the 5% level of significance when the run test statistic is less than 1.965.

<table>
<thead>
<tr>
<th>Sample</th>
<th>Sample size</th>
<th>Run statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Entire dataset</td>
<td>1197</td>
<td>4.37</td>
</tr>
<tr>
<td>Modified dataset</td>
<td>766</td>
<td>0.51</td>
</tr>
</tbody>
</table>

The variance of decibel intensity for both datasets was not significantly different from the Dyer value; however, in both cases, the scintillation index was also considerably more than 1.0. The first finding suggests that the fields at both sites may have reached saturation, but the second finding shows that this is not true. A fuller theoretical understanding of these metrics is warranted.

VI. COMPARISONS WITH PATH INTEGRAL STATISTICS

A. Theoretical quantities

1. Horizontal coherence

FDMWZ theory provides an expression for the structure function $D(\Delta x)$,

$$D(\Delta x) = 2k_*^2 J(\Delta x),$$

(22)

where $k_*$ is the wave number at the sound speed profile minimum and $J(\Delta x)$ is an integral expression derived from a path integral treatment; $J(\Delta x)$ is an integral of decoherence effects along a deterministic eigensray from the source to the receiver.

This integral is implemented numerically in CAFI using expansions and approximations described in Flatte and Stoughton. As discussed in Sec. IV B, the computation for the separations under consideration in this paper is well modeled by the “small separation” approximation, given already in Eq. (18), which is repeated below:

$$D(\Delta x) \approx (\Delta x/L_h)^{3/2},$$

where $L_h$ is the transverse horizontal coherence length. Thus, the CAFI coherence length prediction can be determined directly from the separation at which the coherence falls to $e^{-1/2}$. This predicted coherence length is then compared to the experimentally measured coherence length.

TABLE VI. Summary of estimated transverse horizontal coherence length, scintillation index, log-intensity variance, and decibel-intensity variance over both paths, and the modified sample population size.

<table>
<thead>
<tr>
<th></th>
<th>Kauai-N</th>
<th>Kauai-O</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coherence length (m)</td>
<td>528±152</td>
<td>410±25</td>
</tr>
<tr>
<td>SI</td>
<td>1.53±0.13</td>
<td>1.69±0.16</td>
</tr>
<tr>
<td>$\text{var}(\log(I))$</td>
<td>2.02±0.14</td>
<td>2.01±0.14</td>
</tr>
<tr>
<td>$\text{var}(I_{\text{dB}})$ (dB$^2$)</td>
<td>(6.17±(1.56)$^2$</td>
<td>(6.28)$^2+(1.62)^2$</td>
</tr>
<tr>
<td>Modified sample size</td>
<td>766</td>
<td>766</td>
</tr>
</tbody>
</table>
TABLE VII. CAFI input parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Source depth</td>
<td>810 m</td>
</tr>
<tr>
<td>Carrier frequency</td>
<td>75 Hz</td>
</tr>
<tr>
<td>Adiabatic gradient $\gamma_A$</td>
<td>$1.2 \times 10^{-5}$ s$^{-1}$</td>
</tr>
<tr>
<td>Ref. buoyancy freq. $n_a$</td>
<td>3.0 cph</td>
</tr>
<tr>
<td>Modal bandwidth $j_w$</td>
<td>3.0</td>
</tr>
<tr>
<td>rms displacement $\xi_0$</td>
<td>7.3 m</td>
</tr>
<tr>
<td>Ray launch angle</td>
<td>$12.37^\circ$</td>
</tr>
</tbody>
</table>

2. Scintillation

CAFI models the intensity moments as

$$
\langle I^n \rangle \approx n! \langle I \rangle^n \left[ 1 + \frac{n(n-1)}{2} \gamma \right],
$$

(23)

where $\gamma$ is the “microray focusing parameter.” Using Eq. (15), the CAFI prediction of SI is

$$
SI \approx 2 \gamma + 1.
$$

B. Modeling results

Fluctuation calculations were conducted for two paths intended to represent the paths Kauai-N and Kauai-O. The primary CAFI input parameters are shown in Table VII. The calculations used representative ranges between 2000 and 3000 km for the two scenarios.

CAFI requires range-independent sound speed and buoyancy profiles. Range-independent profiles were generated from the annual Levitus climatology by extracting temperature and salinity profiles every 100 km along each path, converting into sound speed and Brunt–Väisälä buoyancy frequency, and then averaging over the full range to yield the required input profiles. This averaging will smooth out local extremes in the buoyancy profile that might cause enhanced scattering, but is more consistent with the processing applied to the data, which averaged over nearly two full years.

CAFI also requires a profile of the (depth-dependent) sound speed variance. Denoting the random nondimensionalized sound speed fluctuations at depth $z$ as $\mu(z)$, the variance profile is

$$
\langle \mu^2(z) \rangle = \xi_0 n(z) \left[ \frac{1}{c} \frac{dc}{dz} - \gamma_A \right]^2,
$$

(24)

where $\xi_0$ is a rms internal wave vertical displacement, $n(z)$ is the buoyancy frequency as a function of depth, $n_a$ is a reference buoyancy frequency, and $\gamma_A$ is the adiabatic gradient. This variance profile was computed directly from the averaged sound speed and buoyancy frequency profiles using spline interpolation.

CAFI associates fluctuation calculations with deterministic rays launched at user-prescribed angles from the transmitter. For this comparison, the range-dependent code EIGENRAY was used to calculate deterministic eigenrays from transmitter to receiver. The calculations used the same annual Levitus climatology as above, with sound speed profiles extracted again every 100 km along the path. Bathymetry was ignored, except that only upward-going rays at the bottom-mounted transmitter were considered.

For both paths, EIGENRAY could not find wholly refracted–refracted rays from transmitter to receiver: the receivers were slightly too deep. As a compromise, the receiver depth was raised until an eigenray solution could be formed. This solution had a lower turning point directly above the true receiver. Such a representative early-arriving eigenray was chosen for each Kauai-N and Kauai-O path, and the initial launch angle of this ray was input to CAFI. The final CAFI results were not sensitive to small changes (of order $0.5^\circ$) in this initial launch angle.

The averaged sound speed and buoyancy frequency profiles, and the corresponding $\langle \mu^2 \rangle$ profile for both paths are shown in Fig. 12, Fig. 13, and Fig. 14. A summary of CAFI calculations are shown in Table VIII. Statistics for both paths are similar, as might be expected, since these paths have similar lengths over similar regions of the ocean. The CAFI horizontal coherence functions are shown in Fig. 15. A least-squares estimate of $L_h$ based on Eq. (18) yields $L_h = 414.1$ m and $L_h = 366.8$ m for Kauai-N and Kauai-O, respectively. The value $\gamma$ yields via Eq. (23) scintillation indices of approximately 1.47 and 1.32 for the Kauai-N and Kauai-O paths, respectively. For reference, a diagnostic parameter $\phi$ is supplied by CAFI; the path-integral treatment is considered valid when this parameter is less than 1.0. For both runs, this validity condition holds.

Signal fluctuations are characterized in FDMWZ theory as belonging to one of several regimes, depending on the values of a “strength parameter” $\Phi$ and a “diffraction parameter” $\Lambda$. The strength parameter characterizes the average variance of the sound speed fluctuations along the path. The diffraction parameter characterizes the amount of diffraction from sound speed inhomogeneities.
For $\Phi > 1$, the boundary $\Lambda \Phi = 1$ in $\Lambda$, $\Phi$ space separates the “fully saturated” regime from the “partially saturated” regime. In both regimes, using the language of ray theory, the acoustic field separates into micromultipaths. In the partially saturated regime, the micromultipaths are partially correlated, but in the fully saturated regime, the micromultipaths are completely uncorrelated.

The computed values of $\Lambda$ and $\Phi$, and specifically the product $\Lambda \Phi$, predict that both these rays are in the fully saturated regimes.

VII. DISCUSSION

Predictions and measurements show remarkably good agreement, even though the theory, which is based on the ray acoustics paradigm, is not thought to be valid at or below about 100 Hz, nor near caustics. The agreement found in this paper suggests that FDMWZ theory makes accurate predictions of horizontal spatial coherence for deep arrivals at 75 Hz, at least for these scenarios.

Theoretical calculations indicate that the Kauai-O path, which is shorter, should have less coherence (i.e., more scattering) than the Kauai-N path. This surprising result is supported, at least to within a standard deviation, by measure-

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Kauai-N</th>
<th>Kauai-O</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scattering regime</td>
<td>Fully saturated</td>
<td>Fully saturated</td>
</tr>
<tr>
<td>Diffraction parameter $\Lambda$</td>
<td>0.13</td>
<td>0.23</td>
</tr>
<tr>
<td>Strength parameter $\Phi$</td>
<td>8.57</td>
<td>8.35</td>
</tr>
<tr>
<td>$\Lambda \Phi$</td>
<td>1.11</td>
<td>1.92</td>
</tr>
<tr>
<td>Transverse horizontal coherence $L_h$</td>
<td>414.1 m</td>
<td>366.8 m</td>
</tr>
<tr>
<td>Microray focusing $\gamma$</td>
<td>0.24</td>
<td>0.16</td>
</tr>
<tr>
<td>Scintillation index $SI$</td>
<td>$\sim 1.47$</td>
<td>$\sim 1.32$</td>
</tr>
<tr>
<td>$\phi$ (anisotropy)</td>
<td>0.027</td>
<td>0.023</td>
</tr>
<tr>
<td>$\phi$ (inhomogeneity)</td>
<td>$-0.80$</td>
<td>$-0.79$</td>
</tr>
</tbody>
</table>

FIG. 13. Averaged buoyancy profile derived from annual Levitus climatology profiles and averaged over the source–receiver path.

FIG. 14. Sound speed variance profile derived from the averaged sound speed and buoyancy profiles for each source–receiver path.

FIG. 15. A comparison of horizontal coherence models: top, Kauai-N path, bottom, Kauai-O path. The dotted lines represent the model based on empirical data, the solid lines represent the CAFI “small separation” approximation. $\times$=coherence length estimate from data; $\square$=coherence length from CAFI. The shaded region represents approximately the standard deviation error in the empirical model.
ments. One might expect the reverse to be true, based solely on path length. The sound speed variance profiles in Fig. 14 may explain this: there is more intense sound speed variability around 200 to 300 m in the Kauai-O path than in the Kauai-N path. This increased variability would result in more scattering over the Kauai-O path and therefore less coherence at site O. Thus, the path length alone does not provide a full characterization of spatial coherence—path-dependent sound speed fluctuation strength must also be considered.

The empirical measures are not precisely equivalent to the theoretical quantities. Given the individual contributions of each signal processing step, as discussed in Sec. IV A, it is reasonable to suspect that the empirical mutual coherence probably underestimates the true mutual coherence.

It is interesting to consider the consequences of arrival angle wander. By the Van Cittert-Zernike theorem, 44 the mutual coherence is the Fourier transform of the spatial spectrum of an extended incoherent source. Let the angle of arrival of radiation from the source be uniformly distributed between azimuthal angles \( \bar{\theta} - 1/2 \Delta \theta \) and \( \bar{\theta} + 1/2 \Delta \theta \) during the experimental period. The corresponding wave numbers are related by

\[
k = \frac{2 \pi}{\lambda} \sin \bar{\theta}.
\]

Then, to first order in \( \Delta \theta \), the source spatial spectrum has the distribution

\[
B(k) = \begin{cases} 
\left( \frac{\pi \Delta \theta}{\lambda} \right)^{-1}, & k \in \left[ -\frac{\pi \Delta \theta}{\lambda}, k + \frac{\pi \Delta \theta}{\lambda} \right] \\
0, & \text{otherwise},
\end{cases}
\]

where \( \bar{k} = (2 \pi/\lambda) \sin \bar{\theta} \) (valid when \( \Delta \theta \leq 1 \)). The magnitude of the Fourier transform of \( B(k) \) is therefore proportional to

\[
\sin \frac{x \pi \Delta \theta}{\lambda}.
\]

From Eq. (14), there is a measure of \( |\Gamma(x)| \) at \( x = L_h \) equal to \( e^{-1/2} \). Hence, set

\[
\sin \frac{L_h \pi \Delta \theta}{\lambda} = e^{-1/2}.
\]

This is a transcendental equation with a principal root at

\[
\pi L_h \Delta \theta / \lambda = 1.6443,
\]

or

\[
\Delta \theta = \frac{\lambda}{\pi} \frac{1.6443}{L_h}.
\]

The corresponding angular “source widths” are given in Table IX. If arrival angle variability were the only factor contributing to decoherence across the array, the variation in the incident field arrival angle could be attributed to seasonal horizontal sound speed effects. However, since other factors contribute to decoherence, this angular estimate can only serve as an upper bound on the apparent influence of horizontal sound speed effects.

Two investigations into horizontal refraction along the path from the Kauai source to a receiver near Pt. Sur on the California coast (a path length of approximately 3847 km) inferred horizontal refraction angles introduced over that path that were slightly smaller than the “angular width” found above. \(^{45,46}\) Although those analyses involved a different ocean path, the path lengths and ocean regions are similar to those reported here, and consequently the horizontal refraction angle effects might be of a similar order of magnitude in the Kauai-O and Kauai-N data. If this is true, then the spatial coherence at sites N and O due solely to scatter (i.e., the “coherence of a single incident field”) would be slightly better than the measurements reported in Sec. V C, which are estimates containing the influence of both scatter and arrival angle wander.

VIII. SUMMARY

Predictions of transverse horizontal mutual spatial coherence from path integral theory were compared to measurements made on two U.S. Navy receiving arrays during the NPAL experiment in the North Pacific. The source was a 75 Hz bottom-mounted projector moored near the SOFAR channel axis at a depth of 810 m on the northern slope of Kauai. The source transmitted \( m \)-sequences with a period of 27.28 s for an average duty cycle of about 2% over the experiment.

Two receivers at ranges between 2000 and 3000 km had enough SNR after pulse compression and 4-period summation to allow the deep arriving signals to stand out from the background noise on individual channels. This made the direct computation of mutual coherence possible for these two receivers.

The data processing was designed to yield quantities comparable to theoretical single frequency quantities. Time gating was used after pulse compression to isolate partial fields corresponding to deterministic eigenrays, and Fourier coefficients with 2 Hz bandwidth were substituted for single-frequency quantities. A special algorithm was applied to estimate the coherence matrix in the presence of additive ambient noise without incurring the bias of ambient noise coherence. Realizations were averaged over the two-year span of the dataset. Statistics at both receivers were similar: the transverse horizontal coherence length measured about 400–500 m, the scintillation index about 1.5 to 1.6, and the intensity variance about equal to the 5.6 dB Dyer value. There was no significant seasonal trend in the intensity statistics.

Overall, the comparisons between the measurement and theoretical prediction of the second moment were quite good. Computations from CAFI were remarkably close, with predictions of horizontal coherence lengths of about 400 m and a scintillation index of about 1.3 to 1.5. An apparent decrease in the measured coherence on the shorter Kauai-O path was successfully predicted by CAFI. Measurements similar to

\[
\begin{array}{|c|c|}
\hline
\text{Path} & L_h (\text{m}) & \Delta \theta (^\circ) \\
\hline
\text{Kauai-N} & 410 & 1.46 \\
\text{Kauai-O} & 528 & 1.13 \\
\hline
\end{array}
\]
where the signal sequence and overshot’’ across the array, leading to omitted.

ACKNOWLEDGMENTS

We would like to acknowledge discussions with F. Henney, T. Ewart, M. Wolfson, and B. Dushaw. We would also like to thank B. Dushaw for providing the eigenray arrival tracks, and the data for Figs. 1, 5, and 8. This work was supported by ONR code 321.

APPENDIX: UNBIASED COHERENCE MATRIX ESTIMATOR

Let the measured data on channel \( j \) be

\[
x_j(t) = s_j(t) + n_j(t),
\]

where the signal \( s(t) \) is a scintillating phase-encoded \( m \) sequence and \( n(t) \) is additive ambient noise. The Fourier transform of \( x(t) \) is

\[
X_j(\omega) = S_j(\omega) + N_j(\omega),
\]

where \( X(\omega) = \mathcal{F}\{x(t)\} \) is the Fourier transform of \( x(t) \), etc. In the following, it will be convenient to adopt vector-matrix notation. Fourier components at frequency \( \omega \) (henceforth omitted) for each channel are organized into a vector “snapshot” across the array, leading to

\[
x = s + n,
\]

where \( x = [X_1, X_2, \ldots, X_N]^T \) for an \( N \)-sensor array. The data covariance matrix is then

\[
C_{xx} = \langle xx^H \rangle
\]

\[
= \langle ss^H \rangle + \langle nn^H \rangle
\]

\[
= C_{ss} + C_{nn},
\]

where \( \langle \cdot \rangle \) denotes Hermitian transposition and \( \langle \cdot \rangle \) expectation. Equation (A5) follows because the signal and noise processes are independent. If \( x_k \) denotes the \( k \)th-independent realization of the process \( x \), then a standard estimator for the data covariance matrix is the sample covariance matrix, defined as

\[
\hat{C}_{xx} = \frac{1}{K} \sum_{k=1}^{K} x_k x_k^H.
\]

Consider solely the signal covariance matrix \( C_{ss} \). This has elements

\[
[C_{ss}]_{jk} = \begin{cases} 
\sigma_j \sigma_k \rho_{j,k}, & j \neq k, \\
\sigma_j^2, & j = k.
\end{cases}
\]

By definition, \( \sigma_j^2 \) is real and positive, and \( \rho_{j,k} \) is, in general, complex. \( C_{ss} \) is also called the cross-spectral density matrix. Using the transform

\[
|G|_{jk} = |[C]_{jk}| / \sqrt{|C|_{jj} |C|_{kk}},
\]

one obtains a matrix of correlation coefficients. Allowing that \( s(t) \) is the acoustic field measured at sensor location \( r_j \), and using Eq. (7), \( G \) is simply seen to be a matrix notation for the samples of the mutual spatial coherence function magnitude at separations \( r_j - r_k \). \( G \) will therefore be called the coherence matrix, and it contains all the information available regarding the functional shape of \( |G(r_j) - r_k)| \).

Inasmuch as the coherence matrix depends on the signal covariance matrix, it becomes centrally important to develop an unbiased estimator for the signal covariance matrix. Unfortunately, the signal covariance matrix is not available separately: the signal is measured in the presence of noise. The noise vector \( n \) can usually be well approximated by a complex multivariate Gaussian process with \( \langle n \rangle = 0 \) and covariance matrix \( C_{nn} \). This noise covariance matrix possesses an intricate and usually unknown structure. This structured matrix assumes the role of a nuisance parameter, which, in a conventional estimation problem, will have to be estimated separately and removed. This cannot be accomplished for the current problem, but failure to account for this nuisance parameter will result in biased estimates of both the signal covariance matrix and hence the coherence matrix.

An alternative technique for constructing an unbiased estimator for the signal covariance matrix has been suggested by Wong et al.\footnote{Wong et al. 1975} In this technique, two snapshots are considered:

\[
x^{(a)} = s^{(a)} + n^{(a)},
\]

\[
x^{(b)} = s^{(b)} + n^{(b)}.
\]

Snapshots \( a \) and \( b \) are collected at two times \( t_a \) and \( t_b \) far enough apart that the two noise vectors have decorrelated. Then the covariance between these two vectors is \( \delta \Omega_N \). This allows

\[
C_{xx} = C_{ss}.
\]

This situation is shown schematically in Fig. 16.

The structure of \( C_{ss} \) in this situation is no longer the same as Eq. (A8) because \( s^{(a)} \neq s^{(b)} \). Now,

\[
[C_{ss}]_{jk} = \begin{cases} 
\sigma_j \sigma_k \rho(D_{j,k}, \Delta t_{ab}), & j \neq k, \\
\sigma_j^2 \rho(0, \Delta t_{ab}), & j = k.
\end{cases}
\]

where \( \Delta t_{ab} = t_a - t_b \) and \( \Delta r_{jk} = |r_j - r_k| \). The temporal separation between snapshots \( a \) and \( b \) introduces additional decorrelation. In addition, \( C_{ss} \) is, in general, no longer Hermitian.

If the total correlation \( \rho(D_{j,k}, \Delta t_{ab}) \) is approximated by a spatiotemporal separable form \( \rho(D_{j,k}) \rho(D_{t_{ab}}) \), the normalization of Eq. (A9) will recover a coherence matrix con-
therefore becomes appropriate to use instead

\[ G = \mathcal{T}_X(C_{xx}) = \left[ \mathbf{D}^{-1/2} \left( C_{ss} \otimes C_{ss}^* \right) \mathbf{D}^{-1/2} \right]^{1/2}, \]  

(A11)

where

\[ \mathbf{D} = \text{diag}(\text{diag}(C_{ss} \otimes C_{ss}^*)), \]

and \( \mathbf{X} \otimes \mathbf{Y} \) is the Hadamard product (element by element multiplication) between two matrices \( \mathbf{X} \) and \( \mathbf{Y} \). This new transformation eliminates the dependence of the matrix elements on the temporal decorrelation, leaving a matrix containing only spatial coherence terms.

(The matrix \( \mathbf{G} \) calculated this way is not intrinsically Hermitian since \( \mathbf{s}^{(a)} \neq \mathbf{s}^{(b)} \). However, any estimated coherence should be Hermitian since \( |\Gamma(r_j - r_k)| = |\Gamma(r_k - r_j)| \). It therefore becomes appropriate to use instead

\[ \mathbf{G}_{11} = \frac{1}{2} (\mathbf{G} + \mathbf{G}^T). \]  

(A12)

The matrix \( \mathbf{G}_{11} \) is used in subsequent computations.

The technique does not magically eliminate the effect of the additive ambient noise: the sample data covariance matrix based on this technique, for example, will converge at a rate depending on statistical properties of both the signal and the noise. If the SNR is poor, the convergence will be dominated by the influence of the additive noise. However, the sample data covariance matrix will converge to the signal covariance matrix, and so this technique provides unbiased statistics.


29. For example, L. Brekhovskikh and Yu. Lysanov, op. cit., Chap. 10, pp. 208–216.
30 B. Dushaw (private communication).


34 B. Efron and R. J. Tibshirani, *An Introduction to the Bootstrap* (Chapman & Hall, New York, 1993), Sec. 6.2, pp. 45–49. The bootstrap is a computational procedure for estimating the error of an estimator by reusing the sample data. Briefly, let an estimator $\hat{t}$ use all measurements $x$ in a sample $X = \{x_1, x_2, \ldots, x_N\}$. “New” sample sets $X_1, X_2, \ldots$ are generated such that each sample set contains $N$ elements selected randomly with replacement from the original set. Bootstrap replicates of $\hat{t}$ are then computed from each new sample set, and the bootstrap error in $\hat{t}$ is then the standard deviation of the bootstrap replicates.


37 S. M. Flatté, R. Dashen, W. Munk, K. Watson, and F. Zachariasen, Ref. op. cit., Chap. 10, pp. 165–188.


43 S. M. Flatté (private communication, December 2002).


