Ocean Acoustic Tomography: Mesoscale Velocity

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The travel times of acoustic pulses transmitted in opposite directions over a 300-km distance in mid-ocean have been used to measure the fields of sound speed and absolute water velocity with mesoscale resolution. The experiment, which took place west of Bermuda during August and September 1983, consisted of transceivers on two moorings that sent and received signals every 2 hours for 21 days. Ray paths in opposite directions were found to be very nearly reciprocal; effects due to internal waves and mean currents were small. Sum travel times were inverted to obtain sound speed at 2-day intervals, and differential travel times were inverted to obtain absolute current velocity \( \mathbf{u} \). The principal results are (1) range-averaged sound speed perturbation \( \delta C(x) \) decreased over the 21 days by \( 5 \pm 0.2 \text{ m s}^{-1} \) at 1000 m depth, corresponding to a temperature decrease of approximately 1°C; similar (but lesser) changes are measured at other depths. The changes are associated with the first baroclinic mode. (2) The field \( \delta C(x,z) \) determined from expendable bathythermograph (XBT) measurements at the start of the experiment agrees with the field determined using the acoustic data within estimated errors of \( \pm 1.6 \text{ m s}^{-1} \). It is thus possible to obtain range-dependent \( \delta C \) information from acoustic measurements made in a single vertical slice. (3) During the 21 days the range-averaged barotropic velocity mode increased by a factor of 3, while the baroclinic mode decreased by a factor of 2. The estimated error in range-averaged \( \mathbf{u}(x) \) is \( \pm 2 \text{ cm s}^{-1} \) above the main thermocline and \( \pm 1 \text{ cm s}^{-1} \) below, while for range-dependent \( \mathbf{u}(x,z) \) the errors are 5 and 3 cm s\(^{-1}\), respectively. No significant range dependence in \( \mathbf{u}(x,z) \) was found. The baroclinic velocity agrees with geostrophic velocities inferred from the XBT measurements at deployment and aircraft-deployed XBT measurements 11 days after the end of the reciprocal transmissions.

1. INTRODUCTION

The basic physics underlying the concept of ocean acoustic tomography is simple. The travel time of sound along a path in the ocean between a source and a receiver is a function of the speed of sound and of the water velocity along the path. By making transmissions in opposite directions along a path, the effects of sound speed structure and water velocity can be separated. The sum of the oppositely directed travel times is related to the sound speed structure, while the difference in the travel times is related to the absolute currents. An experiment to employ these ideas in the direct measurement of mesoscale ocean currents using transceivers separated by 300 km was conducted during August and September 1983. This paper describes the experiment, the theory by which quantitative sound speed and current velocity fields are obtained from the travel time measurements, and the results.

The technique of ocean acoustic tomography has been under development for a number of years. \textit{Worcester} [1977] measured differential travel times from two drifting ships 25 km apart and inferred a differential water velocity between a single shallow and a single deep ray path. \textit{Munk and Wunsch} [1979] proposed increasing the scale to hundreds of kilometers and provided a comprehensive plan for mapping mesoscale ocean features. \textit{Spiesberger et al.} [1980] confirmed that multipath arrivals over 900 km were stable in time and could be identified using ray theory. \textit{Cornuelle et al.} [1985] described an experiment conducted in 1981 using one-way acoustic transmissions to map the sound speed (temperature) field due to mesoscale eddies. Four sources and five receivers were deployed in a 300 km square centered at 26°N, 70°W. The results were encouraging, but estimated errors were large because of limited travel time resolution. In the two-way, reciprocal transmission experiment described in this paper (RT683), sources with increased bandwidth significantly improved the resolution of rays and the accuracy of the travel time measurements.

This improved accuracy allows the more stringent measurement of differential travel time to obtain absolute current velocity (rather than sound speed only). The depth-independent barotropic component can be measured directly with high precision, as can the depth-dependent baroclinic components. One of the most appealing applications is the direct measurement of relative vorticity [Rossby, 1975]. With a three-transceiver array, the areal averaged relative vorticity can be measured. With five or more transceivers, one can directly measure not only the gradients of the relative vorticity but also its Laplacian and thus attempt to balance the potential vorticity equation [Longuet-Higgins, 1982; \textit{Munk and Wunsch}, 1982]. Long-term tomographic monitoring of ocean basins will enable measurement of gyre scale dynamical quantities and warming or cooling trends in the ocean.

The experiment described here was primarily a feasib-
Fig. 1. Plan view of the 1983 Reciprocal Transmission Experiment. The moorings are referred to as West (31°21'N, 71°47'W), North (32°41'N, 68°58'W), and South (30°N, 68°58'W). South failed on deployment. The three legs of the triangle, going clockwise from west, are referred to as the North, East, and South legs. West is the origin of the x or range coordinate; +x is from West to North. XBT measurements were made along each leg on deployment (days 212–216) and along only the North leg on recovery (day 275). An AXBT survey of the North leg was made on day 250, 11 days after the end of the reciprocal transmissions. The stars at 31°N, 69°30'W, and 29°N, 70°W show where the Local Dynamics Experiment (LDE) and POLYMODE, respectively, were located during 1978–1979.

ity test of the velocity tomography technique which requires two things: submillisecond timing precision because of the small expected differential travel time signals and reciprocity of ray paths in opposite directions. Before the experiment, it had not been demonstrated that autonomous clocks that would remain sufficiently accurate over long periods of time could be practically deployed in the ocean, and it was not known if internal wave and large-scale current velocities would significantly alter a ray path in one direction relative to the ray path in the opposite direction [Sanford, 1974].

The plan of this paper is as follows. Section 2 describes the acoustic instrumentation and signal processing and other environmental measurements. Section 3 describes the acoustic data processing and the travel-time time series. Ray path reciprocity and travel time errors are discussed. Environmental measurements in the form of sound speed fields are presented. The theory for inverting travel time data for fields of sound speed and current velocity is developed in the context of a Kalman filter in section 4. Sound speed and current velocity results are presented in sections 5 and 6. Because there were few independent and simultaneous direct measurements of current velocity with which to compare the acoustically derived currents, the sound speed results are presented in detail to show that all available data in the experiment are consistent. Section 7 consists of a discussion and conclusions.

2. THE EXPERIMENT

The southern edge of the Gulf-Stream recirculation region was chosen as an area of intermediate current velocities suitable for a first test of mesoscale velocity tomography. The Local Dynamics Experiment (LDE) [Owens et al., 1982] and POLYMODE [Pinardi, 1986] of 1978–1979 took place in this area (Figure 1), providing a priori information on the mean flow and mesoscale statistics. Accordingly, two moorings, named North and West, were deployed 300 km apart to the west of Bermuda in approximately 5400 m of water (Figure 1).

On the North mooring there was a transceiver (source and receiver) with a single receiving hydrophone, and a separate receiver with a four-hydrophone vertical array for determining ray arrival angle. On the West mooring there was both a single-hydrophone transceiver and a separate four-hydrophone transceiver. Data from the four-
Fig. 2. (Left) Reference sound speed profile $C_0(z)$. This range-averaged profile was obtained from the North leg deployment XBT smoothly fared with historical data. (Right) Corresponding ray traces. All rays which have been used are plotted. Table 1 identifies them. Note the shallow and deep turning groups of rays as well as the axial ray.

Hydrophone arrays are presented here; there is no significant difference (except higher signal-to-noise) between these and the single-hydrophone data. The acoustic instruments were placed at 1300 m depth near the sound channel axis. Reciprocal transmissions occurred every 2 hours for 21 days, Julian year days 217 to 239, at which time the North source failed. The West source continued to transmit until recovery on day 275, so one-way results are available until then. A third set of instruments on a southern mooring failed shortly after deployment.

Limited source levels (174 dB re 1 µPa at 1 m) make it necessary to use pulse compression techniques to simultaneously achieve adequate signal-to-noise ratio and high travel time resolution. The transmitted signal is coded and spread out in time, and the received signal is then correlated against a replica of the transmitted signal to make it appear as if a single strong pulse had been sent [Spindel, 1979]. Appendix A gives the details of the procedure.

The rms travel time precision $\sigma_\tau$ associated with the correlation peak is

$$\sigma_\tau \geq \sigma_\text{peak} \text{(SNR)}^{-1/2}$$

where $\sigma_\text{peak}$ is the rms width of the peak and SNR is signal-to-noise ratio [Skolnik, 1970; Flatté and Stoughton, 1986]. Typical values of $\sigma_\text{peak}$ and SNR were 5 ms and 20 dB, respectively, yielding a lower bound of 0.5 ms for $\sigma_\tau$, adequate for measuring the expected small differential travel time signals. The pulse width and the corresponding rms travel time uncertainty were a factor of 6 smaller in this experiment than in the 1981 tomography experiment [Cornuelle et al., 1985].

Measurements of mooring position and tilt were made in order to correct the travel times for mooring motion. Mooring position was determined with a long-baseline acoustic navigation system, supplemented with pressure data from a sensor adjacent to the acoustic instrumentation. A self-contained high-frequency acoustic transceiver attached to the mooring interrogated three transponders placed in an equilateral triangle on the seafloor around the base of the mooring. The three round trip travel times and the pressure were used to obtain $x(z)$ of the acoustic instruments (J. Spiesberger, personal communication, 1984).

To obtain the accurate time keeping necessary for velocity tomography, the frequency of the low-power crystal oscillators used as clocks in each instrument was compared every 12 hours with rubidium atomic frequency standards, stable to 2 parts in $10^{10}$. The latter were used only intermittently, due to their high power consumption.

Expendable bathythermograph (XBT) casts were made at $\sim 18$ km intervals to 1800 m depth along each leg of the triangle on deployment days 212–216 (clockwise from North in Figure 1). On day 250 an air-XBT (AXBT) survey of the North leg was made. A total of 45 casts, each to 750 m depth, were made along five lines separated by $\sim 28$ km parallel to and including the North leg. On day 275, XBT casts were made along the North leg.

A reference sound speed profile $C_0(z)$ appropriate for the time at which the equipment was deployed was constructed by smoothly faving an average of deep historical data with each XBT profile along the North leg and then averaging over range (Figure 2). An historical temperature-salinity ($T-S$) relation was used to obtain salinity values corresponding to the XBT values. Sound speed was computed from $z, T, S$ using the Chen and Miller [1977] formula. The sound speed minimum occurs at 1350 m, approximately where the instruments were placed. Thirteen geometric rays traced between the transceivers on each mooring through the $C_0(z)$ field are also shown in Figure 2 and fully identified in Table 1. Because of the shape of the profile, all surface reflected rays are also bottom reflected; these rays are not used. The nearconstant velocity region between 150 and 600 m is associated with the Sargasso Sea $18^\circ$C water. The effect of this region, with no sound speed gradient to turn rays, is to
TABLE 1. Ray Identifiers

<table>
<thead>
<tr>
<th>n</th>
<th>$\theta_R$, deg</th>
<th>$\theta_R$, deg</th>
<th>$z^+$, m</th>
<th>$z^-$, m</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>+8</td>
<td>11.6</td>
<td>11.6</td>
<td>126</td>
</tr>
<tr>
<td>2</td>
<td>-8</td>
<td>-11.6</td>
<td>-11.7</td>
<td>125</td>
</tr>
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<td>3</td>
<td>+9</td>
<td>12.0</td>
<td>-12.0</td>
<td>99</td>
</tr>
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<td>4</td>
<td>+11</td>
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<td>617</td>
</tr>
<tr>
<td>5</td>
<td>-11</td>
<td>-10.8</td>
<td>10.2</td>
<td>737</td>
</tr>
<tr>
<td>6</td>
<td>+12</td>
<td>9.7</td>
<td>9.7</td>
<td>776</td>
</tr>
<tr>
<td>7</td>
<td>-12</td>
<td>-9.6</td>
<td>-9.7</td>
<td>780</td>
</tr>
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<td>8</td>
<td>+13</td>
<td>9.3</td>
<td>-9.3</td>
<td>809</td>
</tr>
<tr>
<td>9</td>
<td>-13</td>
<td>-8.2</td>
<td>8.3</td>
<td>881</td>
</tr>
<tr>
<td>10</td>
<td>+14</td>
<td>7.9</td>
<td>8.0</td>
<td>901</td>
</tr>
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<td>+15</td>
<td>7.4</td>
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<td>925</td>
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<tr>
<td>13</td>
<td>+19</td>
<td>3.5</td>
<td>-3.8</td>
<td>1118</td>
</tr>
<tr>
<td>14</td>
<td>+20</td>
<td>1.2</td>
<td>1.7</td>
<td>1221</td>
</tr>
</tbody>
</table>

The identifier is $\pm n (\theta_R, \theta_R, z^+, z^-)$, where positive (negative) rays depart upward (downward) from the source, $n$ is the total number of upper and lower turning points, $\theta_R$ is the departure angle at the source, $\theta_R$ is the arrival angle at the receiver, and $z^+$ and $z^-$ are the upper and lower turning depths, respectively. The +20 axial ray replaced the +19 ray in the final inversions.

eliminate rays which would otherwise have upper turning depths in this depth interval. This results in two groups of rays: shallow turning rays (< 100 m) and deep turning rays (> 600 m). The most important aspect of Figure 2 is to note how well the entire section is sampled by the rays, especially between 600 and 3500 m.

The predicted arrival pattern (amplitude versus arrival time) for this geometry and $C_0(z)$ is shown in Figure 3. The usual characteristics of mid-latitude sound propagation are evident: weak early arrivals of near-surface-turning rays, later arrivals of deep-turning rays, climax by a strong final, sound channel axis arrival cluster. The WKBJ Green's function technique was used to make this prediction so both ray geometric and diffracted arrivals are included [Brown, 1982]. Departures from this predicted arrival pattern are the basic data to determine departures from $C_0(z)$.

3. OBSERVATIONS

Acoustic Measurements

Daily averaged arrival patterns for the receiver on the North mooring are very stable (Figure 4). Ray arrival peaks can be easily followed from day to day; the amplitudes are roughly constant. The variation in total travel time is due primarily to changes in the source-receiver range because of mooring motion. A 0.75-s change implies a differential horizontal mooring displacement of approximately 1100 m, consistent with the maximum measured mooring displacement.

Figure 5 shows a sequence of reciprocal arrival patterns. Because transmissions start at the same time, sound from one source crosses sound traveling in the opposite direction from the other source. The number of arrivals and the spacing between arrivals are quite similar. The amplitudes, though, vary as much as a factor of 2, and at times an arrival can disappear entirely. This may be a result of storing only the 32 highest peaks in the acoustic instruments (thus losing small ones), or it may be due to ocean changes. Internal waves can cause the basic ray path to split into micromultipaths which then interfere at the receiver, sometimes destructively (rays are predicted to be in the partially and fully saturated regimes) [Flatté, 1983]. Daily averages suppress internal wave-related fluctuations, and the daily averaged arrival patterns for day 223 are nearly the same except for the cluster of arrivals near 204.25 s (from near-axial paths). Interference effects make this part of the pattern very sensitive to small differences. The fact that the amplitudes are different for the two receivers indicates that some nonreciprocity due to currents is present.

In order to perform an inversion of travel time data, each arrival must be associated with a particular ray path. This identification is achieved in part by comparing predicted and measured receptions in the time domain. In Figure 3 the daily average for day 217 is compared with a range-independent WKBJ prediction using $C_0(z)$. The correspondence is excellent, except for the final cluster of near-axial arrivals (the WKBJ method is not valid near ray turning points and thus near the sound channel axis).

![predicted vs measured](image)

Fig. 3. Comparison of the predicted and measured arrival patterns. The measured arrival pattern is an average for day 217. The predicted arrival pattern was calculated from $C_0(z)$ using the WKBJ method of Brown [1982]. Geometric arrivals are labeled $\pm n (z^+)$ as in Table 1. The peaks connected by dashed lines are the ray arrivals actually used.
The small differences between the predicted and measured patterns are due not only to the differences between the reference and actual sound speed and water velocity fields present (the signal) but also to absolute range uncertainty and mooring motion effects. The arrivals which will be used in subsequent analysis are connected by dotted lines; they are fully identified in Table 1.

With this preliminary identification of the arrivals, the data were processed to produce time series of travel time and arrival angle \((r, \theta)\) for each of the different ray arrivals. Tracking the raw data in time to produce the time series was greatly facilitated by the arrival angle information. These time series were then corrected for mooring motion, mooring tilt, and clock drift (Appendix B). The identification is confirmed by a comparison of measured and predicted arrival angles (Figure 6). In Figure 6, all the corrected \(r, \theta\) data for the receptions on day 217 are plotted on top of the predictions in the \(r, \theta\) plane.
Fig. 6. Ray arrivals plotted against travel time and arrival angle. The predicted arrivals are plotted as open circles connected by the smoothly drawn solid curve. The measured one-way arrivals for the 10 receptions on day 217 are plotted as points, with size proportional to SNR. Geometric arrivals are labeled ± n as in Table 1. SRBR refers to surface reflected, bottom reflected rays.

The predictions and the measurements are in very good agreement. Note particularly the separation in angle of the even numbered rays which arrive at nearly the same time. Travel time fluctuations due to noise and internal waves are small on this scale, but the arrival angle fluctuations are relatively large (∼0.5° rms), due almost entirely to the lack of measurement precision (Appendix B), rather than ocean fluctuations.

Next, one-way, sum, and difference series were formed. These were smoothed and interpolated by a digital low-pass filter implemented by convolving the data with a cosine-bell filter window with a total width of 4 days, corresponding to a 3-dB cutoff frequency of 0.5 cpd or a period of 2 days. High-pass time series were formed by subtracting the low-pass series from the original series.

For each of the 13 rays, the sum travel time time series increased by ∼120 ms between days 220 and 234, with little change before and after this period (Figure 7). The increase in all the travel times indicates a cooling trend throughout the water column.

The differential travel time time series fall into two groups, corresponding to shallow and deep turning rays (Figures 8 and 9). The separation of the data into two groups is a qualitative indication of velocity shear in the water column. The separation between the groups decreases with time corresponding to decreasing shear. All the data are positive suggesting a barotropic current in the -x direction. All the differential travel time data become more positive with time, corresponding to an increasingly negative barotropic velocity.

Reciprocity of Ray Paths

The high-frequency correlation of travel times in opposite directions can be used as a measure of ray path reciprocity, i.e., how close the ray paths in opposite directions are to each other. If the ray paths are close in the sense of internal wavelength scales, the correlation will be high. This is the case for this experiment. Table 2 gives the relevant statistics. After removing variance attributable to residual mooring motion, ambient noise, and interpolation error, the corrected high-frequency (> 0.5 cpd)

Fig. 7. The sum travel time time series for all 13 rays used. The solid curves show the low pass (T > 2 days) versions of the series.

Fig. 8. Same as Figure 6, except differential travel time time series. The axial ray is not used.
variance of the one-way travel times \( \langle T^2 \rangle \) is 10.6 ms², while the variance of the differential travel times \( \langle (T^2 - \overline{T}^2)^2 \rangle \) is 0.4 ms² (average of all rays) [Howe, 1986]. The correlation coefficient \( \langle T^2 \rangle \langle T \rangle / \langle T^2 \rangle \) is 0.92. This means that the oppositely directed ray paths are sufficiently close spatially to see nearly the same internal wave field. Internal wave and large-scale shears do not significantly perturb the reference ray path. For the purpose of making low-frequency inversions, the paths are identical in both directions.

Furthermore, this means that differential travel times can be measured with a higher precision than one-way (or sum) travel times since internal wave induced fluctuations largely cancel. The magnitudes of these high-frequency variances can also be interpreted in terms of internal wave strength. This is done in detail by Flatté and Stoughton [1986] and Stoughton et al. [1986].

**Low-Frequency Travel Time Errors**

Table 3 summarizes the low-frequency travel time errors. Part of the low-frequency error variance is made up of high-frequency noise, here including internal wave fluctuations, ambient acoustic noise, and interpolation error. This part is reduced in low-pass filtering by the effective number of points within the filter window, usually about 12. A constant error is added to represent all other low-frequency errors, including clock errors and nonlinearities [Spiesberger and Worcester, 1983]. Typical rms errors \( \epsilon_{LF} \) for the sum and difference travel times are

<table>
<thead>
<tr>
<th>Travel Time</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>One-way travel time variance, ms²</td>
<td>( \langle T^2 \rangle )</td>
<td>10.6</td>
</tr>
<tr>
<td>Differential travel time variance, ms²</td>
<td>( \langle (T^2 - \overline{T}^2)^2 \rangle )</td>
<td>0.4</td>
</tr>
<tr>
<td>Correlation coefficient</td>
<td>( \langle T^2 \rangle \langle T \rangle / \langle T^2 \rangle )</td>
<td>0.92</td>
</tr>
</tbody>
</table>

**Table 2. High-frequency (>0.5 cph) Travel Time Variances**

<table>
<thead>
<tr>
<th>Travel Time</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>High-frequency internal wave and</td>
<td>( \epsilon_{HF} )</td>
<td>25.0</td>
</tr>
<tr>
<td>measurement noise, ms²</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nonlinearity and clocks, ms²</td>
<td>( \epsilon_{const} )</td>
<td>0.9</td>
</tr>
<tr>
<td>Low-frequency error variance, ms²</td>
<td>( \epsilon_{LF} )</td>
<td>3.0</td>
</tr>
<tr>
<td>rms low-frequency error, ms</td>
<td>( \epsilon_{LF} )</td>
<td>1.7</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Sum</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \epsilon_{HF} )</td>
<td>25.0</td>
<td>0.6</td>
</tr>
<tr>
<td>( \epsilon_{const} )</td>
<td>0.9</td>
<td>0.2</td>
</tr>
<tr>
<td>( \epsilon_{LF} )</td>
<td>3.0</td>
<td>0.3</td>
</tr>
<tr>
<td>( \epsilon_{LF} )</td>
<td>1.7</td>
<td>0.5</td>
</tr>
</tbody>
</table>

1.7 and 0.5 ms, respectively. Error bars for two low-pass differential travel time series show the error fluctuations one would expect (Figure 9): smaller where more data are present, larger where less data are present, and increasing at the ends where the low-passed data are biased because of end effects and/or to account for extrapolation.

**XBT Measurements**

Figure 10 shows the perturbation sound speed field relative to \( C_0(x) \), determined from the 18 XBT profiles taken along the North leg during deployment. The profiles were first linearly interpolated in \( x \) to 21 equally spaced stations and then contoured. There are two cold (perturbations \( \delta T \) and \( \delta C < 0 \)) eddies at each end of the leg with a warm one in the center. The XBT sections along the other two legs of the triangle were combined with the North leg data to produce a map of \( \delta C \) at 700 m (Figure 11). There is a cold eddy \( \delta C = -15 \text{ m s}^{-1} \delta T = -3^\circ \text{C} \) in the vicinity of the south mooring, decaying smoothly to the values shown at each end of the North leg section. This implies a geostrophic velocity \( u_x \) along the North leg from North to West, i.e., in the \(-x\) direction. Assuming a linear gradient from the South mooring to the North leg, \( u_x \) is approximately \(-7 \text{ cm s}^{-1} \) at 300 m relative to 1000 m.

The AXBT survey made on day 250 gives an average perturbation along the path at 700 m of \(-3 \text{ m s}^{-1} \) (Figure 12). The range-averaged geostrophic velocity \( u_x \) along the path at 300 m relative to 1000 m is approximately \(-10 \text{ cm s}^{-1} \), in the same direction as implied by the initial XBT data.

The recovery XBT field measured on day 275 is nearly range-independent (Figure 13). The sound speed perturbation is \(-7 \text{ m s}^{-1} \) at 1000 m (the depth of maximum amplitude of the first baroclinic sound speed mode) and \(-4 \text{ m s}^{-1} \) at 700 m.

**Table 3. Low-frequency (<0.5 cph) Travel Time Errors**

Fig. 10. \( \delta C(x,z) \) in m s\(^{-1} \) on day 216 determined by XBT.
4. Theory

In this section we show how the desired fields of sound speed and current velocity are obtained from the travel time measurements.

Acoustic Forward Problem

In the geometric optics approximation the travel time along a ray path $\Gamma_i$ in the presence of a current $u(x,t)$ is

$$T_i(t) = \int_{\Gamma_i} \frac{ds}{C(x,t)+u(x,t) \cdot \tau}$$

(1)

where $\tau$ is a unit vector tangent to the ray and $s$ is arc length. The travel times $T_i$ along the $i$ propagation paths $\Gamma_i$ are the basic data in acoustic tomography. The sign of $u \cdot \tau$ depends on the direction of propagation, and the travel times in opposite directions differ due to the effects of currents.

The travel time path integral (1) is nonlinear in the propagation velocity. Defining a reference sound speed field $C_0(x)$, one has

$$C(x,t) = C_0(x) + \delta C(x,t)$$

Fig. 11. $\delta C(x,y)$ in m s$^{-1}$ at 700 m from the XBT, days 212–216 (clockwise starting at North).

Fig. 12. $\delta C(x,y)$ in m s$^{-1}$ at 700 m from the AXBT survey on day 250.

$$\delta T_i^+ = T_i^+ - T_{0i} = -\int_{T_{0i}}^{T_i} \frac{\delta C(x,t)+u(x,t) \cdot \tau}{C_0^2(x)} ds$$

$$\delta T_i^- = T_i^- - T_{0i} = -\int_{T_{0i}}^{T_i} \frac{\delta C(x,t)-u(x,t) \cdot \tau}{C_0^2(x)} ds$$

The superscript plus (minus) refers to propagation in the + (−) $x$ direction; $\tau$ is negative in the second case since propagation is in the −$x$ direction (the integrals are evaluated in the positive direction). The $i$th reference ray path $\Gamma_{0i}$ is determined in the reference ocean $C_0(x)$. The corresponding reference travel time is

$$T_{0i} = \int_{T_{0i}} ds$$

Forming sums and differences gives

$$s_i = \frac{1}{2} (\delta T_i^+ + \delta T_i^-) = -\int_{T_{0i}}^{T_i} \frac{\delta C(x,t)}{C_0^2(x)} ds$$

$$d_i = \frac{1}{2} (\delta T_i^+-\delta T_i^-) = -\int_{T_{0i}}^{T_i} \frac{u(x,t) \cdot \tau}{C_0^2(x)} ds$$

The problem has separated; $s_i$ is linearly related to the sound speed perturbation $\delta C$, while $d_i$ is linearly related to the water velocity $u \cdot \tau$ along the ray path.

Lateral deviation of the rays in this experiment is small and can be neglected [Munk et al., 1980]. Furthermore, the reference state is taken to be range-independent. With these assumptions, the path integral can be evaluated in terms of $x$ between $x_1 = 0$ and $x_2 = R_0$, the reference $x$ locations of the transceivers. If the actual range $R = R_0 + \delta R$, $s_i$ must be corrected by an amount

$$\frac{\partial s_i}{\partial R} \delta R = \frac{1}{2} \left[ \frac{\cos \theta_i(x_1)}{C_0(x_1)} + \frac{\cos \theta_i(x_2)}{C_0(x_2)} \right] \delta R$$

where $\theta_0$ is the angle that the ray, represented by $z = z_{0i}(x)$, makes with the horizontal [Crommelin et al., 1985]. This assumes that the plane wave approximation is valid over $\delta R$. The term $R_0$ can be thought of as the absolute range between mooring anchors and $\delta R$ the navigation error. The expressions for $s_i$ and $d_i$ are then
Fig. 14. Quasi-geostrophic modes $Z_j$ and $Z_j^u$, $j = 0, 1,$ and 2, for sound speed perturbation $\delta C$ and current velocity $u$, respectively. The Rossby radii of deformation are 2948 km, 43 km, and 17 km.

\begin{align}
\frac{s_j}{\delta \tau} &= \frac{1}{2}(T_i - T_i) - T_B \\
&= -\int_0^R \frac{\delta C(x, z, t)}{C_0^2(z)} \frac{dx}{\cos \theta} + \frac{\partial s_j}{\partial R} \delta R \\
\frac{d_j}{\delta \tau} &= \frac{1}{2}(T_i - T_i) - \int_0^R u(x, z, t) \frac{dx}{C_0^2(z)}
\end{align}

where $d_j$ has been replaced by $\frac{dx}{\cos \theta}$ and $u \cdot \tau$ by $u \cos \theta$, where $u$ is the horizontal component of velocity. Note that $z = z_0(x')$.

**Ocean Model**

Equations (2) and (3) constitute the acoustic forward problem: Continuous $\delta C$ and $u$ fields are used to predict a finite data set $s_j$ and $d_j$. The corresponding inverse problem uses the data to estimate the continuous fields. The latter is an infinitely under-determined problem because values of the continuous $\delta C$ and $u$ fields must be specified at all $x$ and $z$ using only a finite amount of data. To proceed, the continuous fields $\delta C$ and $u$ are therefore parameterized with a finite number of discrete parameters by a model that is oceanographically meaningful. This procedure has three advantages: The parameterization can be very efficient, the inverse solution can become more accurate because the model injects our oceanographic knowledge concerning permissible structure of the fields (reducing the degrees of freedom), and the procedure can be used to indicate data-model consistency.

The quasi-geostrophic spectral ocean model used is described by Cornuelle [1983] (the vertical structure part is reviewed in Appendix C; see also Flierl [1978]). The quasi-geostrophic sound speed modes $Z_j^c$ and current modes $Z_j^u$ are plotted in Figure 14 for $j = 0, 1,$ and 2. Mode 0 corresponds to the barotropic mode and is near zero for $\delta C$ and nearly depth-independent for $u$. (The departures from exact depth independence are due to the free surface boundary condition.) Mode 1 corresponds to the first baroclinic mode; for sound speed it reflects the stretching and compression of the main thermocline, peaking at 1000 m. Baroclinic velocity shears are also highest here. Mode 2 shows increasingly more structure, but its relative importance in terms of expected energy is less. Higher modes with still less expected energy have not been used.

Additional vertical basis functions for sound speed have been found to be necessary to construct consistent inverse solutions. There are at least three plausible reasons for this: (1) Surface forcing (wind shear and heat flux) produces a surface mixed layer not described by the quasi-geostrophic modes. Additional modes are especially necessary when fitting the shallow XBT data to the model; it is not as important for the acoustic data because none of the rays used propagate at depths less than 100 m. (2) The original deep reference profile was constructed from historical data (primarily from 1966). The actual deep sound speed profile can therefore be significantly different due to long-term variability, and some deep gyre scale variation is needed to account for this effect, which can be as large as 0.5 m s$^{-1}$ (0.1°C). (3) Uncertainty in calculating the reference sound speed profile $C_0(z)$ from $z, T, S$ is present. The algorithm of Chen and Millero [1977] was used with errors taken to be the difference between this algorithm and the one given by Del Grosso [1974]. This difference increases roughly monotonically with depth: from $\sim 0$ m s$^{-1}$ at the surface to $\sim 0.6$ m s$^{-1}$ below 3000 m (Chen and Millero is faster).

To model these effects, additional triangular vertical basis functions (suggested by B. Cornuelle) were centered at 10 selected depths: 0, 100, 300, 600, 1000, 1500, 2000, 3000, 4000, and 5250 m ($j = 3$ to $N_f^2 = 13$). Each triangle function is unity at its selected depth and zero at adjacent depths. This parameterization is essentially equivalent to layers of constant sound speed but avoids sound speed discontinuities at the layer interfaces since a weighted sum of the triangular modes is equivalent to linear interpolation between the selected depths. The top two triangle functions model surface effects; the remaining functions model climate and algorithm effects. The latter
trianles (300—5250 m) are taken to be time- and range-independent.

A truncated Fourier series is chosen to represent the horizontal structure. The Fourier series is periodic over a domain \( L \) which is large (600 km) compared to the region in which data are available (300 km) in order to avoid enforcing periodicity within the experimental domain. The high wave number cutoff is \( 2\pi \left( N_p = 4 \right) / L \), or 1 cycle/150 km, adequate to resolve mesoscale eddies.

Summarizing, the sound speed and current fields are represented by

\[
\delta C(x,z,t) = \sum_{i=0}^{N_F} \sum_{j=0}^{N_S-1} \left[ A_{ij}^f(t) \cos \left( \frac{2\pi i x}{L} \right) + B_{ij}^f(t) \sin \left( \frac{2\pi i x}{L} \right) \right] Z_j^f(z) \]

\[
u(x,z,t) = \sum_{i=0}^{N_F} \sum_{j=0}^{N_S-1} \left[ A_{ij}^u(t) \cos \left( \frac{2\pi i x}{L} \right) + B_{ij}^u(t) \sin \left( \frac{2\pi i x}{L} \right) \right] Z_j^u(z) \]

The \( A \) and \( B \) are the desired model parameters, and \( N_S = 3 \). Reordering the \( x \) dependence and the indexing gives

\[
\delta C(x,z,t) = \sum_{i=0}^{N_F} \sum_{j=0}^{N_S-1} a_{ij}^f(t) X_i(z) Z_j^f(z) = \sum_{j=0}^{M_C} m_j^f(t) P_j^f(x,z) \]

(4)

where, for instance,

\[
X(x) = \left[ 1, \cos \frac{2\pi x}{L}, \sin \frac{2\pi x}{L}, \cos \frac{4\pi x}{L}, \sin \frac{4\pi x}{L}, \ldots, \sin \frac{2\pi N_F x}{L} \right]
\]

with a total of \( N_F = (2N_F+1) \) terms and

\[
m^C = [A_0^B, B_0^B, A_0^u, B_0^u, \ldots, A_{N_F}^B, B_{N_F}^B, A_{N_F}^u, B_{N_F}^u, \ldots, B_{N_F}^C ]
\]

with a total of \( M_C = N_F N_{S-1} \) terms. Similarly, for \( u \),

\[
u(x,z,t) = \sum_{i=0}^{N_F} \sum_{j=0}^{N_S-1} a_{ij}^u(t) X_i(x) Z_j^u(z) = \sum_{j=0}^{M_u} m_j^u(t) P_j^u(x,z) \]

(5)

The fields are represented by linear combinations of the known basis functions.

Inverse Problem

The ocean model can be incorporated in the acoustic forward problem by substituting (4) and (5) into (2) and (3), giving

\[
s_i = -\frac{R}{\int_0^R \sum_{j=0}^{M_u} \frac{m_j^f(t)}{C_j^f(z)} \frac{dx}{\cos^2\gamma} + \frac{\partial s_i}{\partial R} \delta R
\]

(6)

Given the model parameters \( m_j^f \) (and \( \delta R \)), the travel times can be calculated. The inverse problem is to obtain estimates of the \( m_j^f \) (and \( \delta R \)) given travel time measurements, \( s_i \) and \( d_i \).

Rewriting equation (7), with implicit time dependence,

\[
d_i = \sum_j m_j G_{ij} \int \frac{P_j^f(x,z)}{C_j^f(z)} dx
\]

(8)

where

\[
G_{ij} = \frac{\partial d_i}{\partial m_j}
\]

is the matrix of partial derivatives giving the dependence of the \( i \)th datum on the \( j \)th model parameter. In matrix notation, equation (8) can be written

\[
\mathbf{d} = \mathbf{G}^T \mathbf{m}^u + \epsilon^u
\]

where \( \epsilon^u \) has been added to account for errors in the measurements or inadequacies in the model. Similarly,

\[
\mathbf{s} = \mathbf{G}^T \mathbf{m}^C + \epsilon^C
\]

where \( \delta R \) has been incorporated as another model parameter in \( \mathbf{m}^C \).

We wish to obtain estimates \( \hat{\mathbf{m}} \) of the true model parameters \( \mathbf{m} \). In general, the problem is mixed determined or underdetermined: There are more model parameters than data. To obtain a unique inverse of \( \mathbf{G} \), we use the weighted, damped, least squares procedure which minimizes the squared Euclidean norm of both the data residuals \( ||\mathbf{d} - \hat{\mathbf{d}}||_2^2 \) and the estimated model parameters \( ||\hat{\mathbf{m}}||_2^2 \). This method is outlined by Lawson and Hanson [1974] and Cornuelle [1983] and also by Aki and Richards [1980] and Menke [1984]. Appendix D describes the procedure in detail.

A major weakness of this method is that the data and model parameters are assumed uncorrelated from day to day. However, model parameters representing mesoscale features in the ocean have finite correlation time scales, while other model parameters such as \( \delta R \) and the amplitudes of the deep sound speed basis functions have very long time scales. Therefore the method is modified by incorporating a Kalman filter (B. Cornuelle, personal communication, 1985) which combines past and present data to better estimate model parameters (Appendix E).

5. Sound Speed

Preliminaries

The sound speed inversion process was started by specifying a reference state \( C_0(z) \) and a priori variances \( W_0 \) of the model parameters about that reference state. \( C_0(z) \) was taken to be the profile constructed by combining the XBT deployment data with the deep historical data (Figure 2). Relative variances of the quasi-geostrophic model
parameters were obtained from a three-layer eddy-resolving general circulation model (courtesy of W. Holland). Horizontal correlation length scales were approximately 100 km. The absolute magnitudes were fixed by scaling so that the a priori variance at 700 m corresponded approximately to the temperature variance measured at 700 m (0.41°C) during the LDE [Bryden, 1982, Table 1]. The two shallow triangle basis functions, representing the surface layer in an ad hoc way, were also given correlation length scales of 100 km; the absolute magnitudes were set to permit 5 m s⁻¹ (1°C) variations at the surface and 3.5 m s⁻¹ at 100 m. The deep triangle basis functions representing climatology were permitted 1 m s⁻¹ (0.2°C) variations. Results are only weakly dependent on these a priori assumptions.

The a priori rms sound speed perturbation associated with \( W_0 \) is plotted in Figure 15a as a function of depth. The expected variance in the total \( \delta C(x,z) \) is independent of \( x \). The right most curve in Figure 15b shows the expected possible variation in the range-averaged

\[
\delta C(z) = \frac{1}{R} \int_0^R \delta C(x,z) \, dx
\]

The curves in Figure 15a and 15b look similar, although the latter has somewhat smaller magnitude than the former; this difference is the expected amount of purely range-dependent versus range-independent energy. Both curves show a high uncertainty ~5 m s⁻¹ due to surface effects, a peak of ~6 m s⁻¹ at ~1000 m due to thermocline stretching, and a 1 m s⁻¹ uncertainty in the deep ocean. These curves show clearly the amount of variation the solution may have as a function of depth. The variation and the way in which it is correlated over range, depth, and time effectively constrain the inverse problem and, with the least squares criteria, serve to isolate a particular solution.

An iteration of the inverse consists of the following steps: (1) The reference sound speed field and a priori variances are specified, (2) the ocean model is fit to the deployment XBT perturbation field, and (3) these model parameters and uncertainties are combined with the day 218 and subsequent acoustic data, using the Kalman filter, to obtain estimates of model parameters and uncertainties for the entire time series. Several iterations were necessary to stabilize the axial (ray) arrival. After the first iteration, a new \( C_1(z) \) was computed

\[
C_1(z) = C_0(z) + \delta C_1(z)
\]

where \( \delta C_1(z) \) consists of the final (day 238) estimates of the time-independent deep triangle amplitudes (Figure 16). A new predicted arrival pattern and ray geometry were computed using this \( C_1(z) \) and a new range estimate \( R_1 = R_0 + \delta R_1 \). It was found that new axial rays appeared and the ray which had been initially identified with the measured axial arrival was no longer the final arrival; it arrived 30 ms before the new predicted final ray arrival. This is a result of the fact that slight changes in the near-axial ocean create or annihilate rays in the final arrival, a nonlinear effect outside the range of validity of this linearized perturbation treatment. To proceed, the measured axial arrival is assumed to correspond to the last predicted ray arrival (arbitrarily with positive launch angle). A large error of 10 ms is assigned to it to reflect additional uncertainties. To stabilize the axial ray, the procedure outlined above was repeated using the last \( C_1(z) \) as a new reference state (but using the same \( W_0 \)) until the axial ray in the \( C_1(z) \) field remained the axial one in the new \( C_{n+1}(z) \) field. Three iterations were required to achieve this stability.

The estimates of the time-independent parameters are refined as new data with independent noise realizations are assimilated by the Kalman filter. In the fourth iteration, \( \delta R \) changes from the initial 0±100 m to 0±34 m on day 218 when the first acoustic data are combined with the

Fig. 15. A priori expected variations in \( \delta C \) as functions of depth (expressed as rms). (a) Range average of the expected variance of the total sound speed perturbation \((R^{-1} \int \delta C^2(x,z) \, dx)\). (b) Expected variance of the range-averaged sound speed perturbation \((R^{-1} \int \delta C(x,z) \, dx)\). The curve labeled \( W_0 \) is the priori variances associated with \( W_0 \), which includes large values (1 m s⁻¹) for the deep triangle functions. The variances of the time-dependent model parameters in \( W_4 \) are the same as in \( W_0 \); the variances, though, of the time-independent parameters (range and deep triangles) in \( W_4 \) have been reduced from the \( W_0 \) values by means of the iterations described in the text. The cross at 700 m shows the LDE measurement.

Fig. 16. Changes \( \delta C_1(z) \) in m s⁻¹ made to the reference state \( C_0(z) \) to stabilize the axial arrival. \( C_1(z) = C_0(z) + \delta C_1(z) \) was the reference state for the final inversions (pass 5).
Fig. 17. Estimates (solid) of the time-independent (t = ∞) model parameters δR in m and δCₚ (3000 m) in m s⁻¹ as functions of time (assimilating new data in pass 4). The δC is relative to C₀. The dashed lines are the estimated error bars.

XBT data and finally to 38 ± 11 m on day 238 (Figure 17). The XBT data are clearly important in helping to determine δR and reducing its uncertainty. In a similar fashion, the estimated uncertainty of the amplitude of the 3000 m triangle function starts at ±1 m s⁻¹, and has a final value on day 238 of ±0.03 m s⁻¹.

In the last (fifth) iteration the reduced uncertainties in the time-independent parameters from the previous (fourth) iteration were incorporated in order to make the best estimate of the time-dependent parameters. W₀ was modified by replacing the relatively large original uncertainties of the time-independent model parameters with the smaller uncertainties estimated from day 238, iteration 4. The difference between the original and new a priori variances W₀ and W₄ can best be seen in Figure 15; the deep variances are now smaller (except in the deepest layer where no rays penetrated).

**Results**

The day 216, XBT measurements can now be compared with estimates of the δC(x, z) field on day 218 obtained in several different ways. Contour plots of four δC fields and their associated estimated errors are shown in Figure 18. Figure 18a shows the result of fitting the XBT deployment measurements to the ocean model using the new a priori variances W₄. The rms errors on individual XBT measurements were approximately 1 m s⁻¹ (instrument 0.5 m s⁻¹ or 0.1°C, nonsimultaneity of measurement 0.5 m s⁻¹, and, as functions of depth, internal wave noise ~0.5 m s⁻¹ and sound speed algorithm error ~0.2 m s⁻¹). The rms uncertainty in the smooth field is typically 0.4 m s⁻¹ at 1000 m. The range average of this error map is shown in Figure 19a. The similarity of the smooth field to the raw field (Figure 10) shows that the ocean model adequately represents the data: the differences between the two are less than the estimated errors in the measurements. Figure 18b is the result of combining the model parameters and uncertainties from Figure 18a with the acoustic travel times for day 218, using the Kalman filter procedure. This method is equivalent to combining the day 218 acoustic data with all the raw XBT data, with errors increased by 0.5 m s⁻¹ to account for the time difference. The δC fields in Figures 18a and 18b look essentially the same, although the error has increased to ~0.8 m s⁻¹ at 1000 m due to the XBT information fading with time. The data residuals, the measured travel times minus the estimated travel times through field in Figure 18b, are less in magnitude than the estimated travel time errors showing the two independent measurements are consistent with each other.

A single inversion using W₄ and day 218 acoustic data only was made to see how well acoustic data alone, together with a good range estimate, could estimate the field (Figure 18c). This field looks very similar to the ones in Figures 18a and 18b, showing the same ~4, +3, −2 m s⁻¹ variation with range at 1000 m. The errors are larger, ~1.6 m s⁻¹, due to the complete absence of XBT data, but they are still smaller than the assumed a priori expectations of mesoscale variability, ~6 m s⁻¹ at 1000 m. This is most obvious by comparing the curves labeled W₄ and 1 in Figure 19a. Much of the a priori uncertainty in the field is reduced by the acoustic data. The range dependence of the estimated field is significant because variations within it are larger than the estimated errors. This result is possible only because accurate estimates of the range and, to a lesser extent, deep triangle amplitudes have been used. A similar case was run assuming a good range estimate was not available; that is, W₀ rather than W₄ was used with the day 218 acoustic data. The field smeared out, range dependence was not resolved, and the errors increased to 2.5 m s⁻¹ (Figure 18d). These results show that with a good range estimate range-dependent δC(x, z) can be measured through a single slice of ocean using acoustic data.

The sound speed perturbation field as a function of time
Fig. 18. Comparing XBT and acoustic measurements. The sound speed perturbation field in m s\(^{-1}\) is on the left and the estimated error field is on the right. (a) The measured XBT field on day 216. (b) The acoustic data on day 218 are combined with the XBT field (see text). (c) Acoustic data only with a good range estimate are used to obtain the field. (d) Acoustic data only are used with large range uncertainty.

is shown in Figure 20. The first panel again shows the smoothed XBT measurements on day 216. Following this are the results incorporating the acoustic data every 2 days until day 238. The general trend is clear; the entire field is becoming more negative (colder water) and more uniform and smoother with range. By day 226, the XBT influence has completely died out because of the 10-day time constant \(\tau\). As mentioned before, the surface values decay back to zero (and uncertainties increase to a priori values) because there is no acoustic information above 100 m.

On day 238 the sound speed at 1000 m goes from \(-6\) m s\(^{-1}\) at the West mooring to \(-2\) m s\(^{-1}\) at the North mooring in a linear fashion. At 700 m it goes from \(-3\) to \(-1\) m s\(^{-1}\). In a qualitative sense these can be compared with the AXBT survey taken 12 days later on day 250, shown plotted at 700 m in Figure 12; the same magni-
Fig. 19. As for Figure 15 except for three new curves in each panel. The curve labeled XBT is the estimated error of the deployment XBT measurements. The curve labeled I (for independent) shows the error when only acoustic data for a single day are used to obtain the field. The curve labeled K shows the error when data (day 234) are used within the framework of the Kalman filter.

Fig. 20. The sound speed perturbation field as a function of time. The first panel, day 216, shows the field as measured by XBT. The subsequent panels, days 218–238, are obtained by assimilating acoustic data; by day 226 all influence of the XBT has disappeared.
is reduced to 0.20 m s\(^{-1}\); the very shallow and very deep errors remain the same. When XBT only are used to calculate a range-averaged \(\delta C(z)\), the errors are slightly larger, 0.25 m s\(^{-1}\) at 1000 m. The fact that the range-averaged XBT and acoustic errors are nearly the same illustrates the spatial averaging power of the acoustic method and shows graphically how 13 acoustic data compare with 18 XBT profiles. The XBT measure the surface waters well, and the error is small there. Most of the deep error is due to uncertainty in the range- and time-independent triangle modes.

Changes in time of \(\delta C(z)\) are clearly displayed by plots of the range-averaged baroclinic mode amplitudes (Figure 22). In the notation of (4), the range-averaged amplitude for mode \(j\) is

\[
\frac{1}{K} \int_0^L \sum_{i=1}^{N_x} a_j^i X_i(x)\,dx
\]

Most of the cooling trend in the sound speed profiles is associated with the amplitude growth of the first baroclinic mode. Toward the end of the experiment, the second baroclinic mode becomes significant. The errors on both

6. CURRENT VELOCITY

The inversion of differential travel times for current velocity was simpler than the inversions of sum travel times for sound speed. The ocean model for currents consisted of the first three quasi-geostrophic modes with the range dependence given by the sines and cosines as before. Absolute range between transceivers does not enter into the velocity problem. The triangle modes were not used because the small sizes of the differential travel times (~10 ms) relative to the measurement precision (~0.5 ms rms) did not warrant the added complexity; the data were consistent with the simpler model. The a priori variances for the quasi-geostrophic velocity modes were determined in the same way as for sound speed, i.e., from Holland's three-layer general circulation model and from data. Correlation length scales were ~100 km; absolute magnitudes were fixed by scaling so that the a priori variance at 500 m corresponded approximately to the velocity variance measured at 500 m (100 cm\(^2\) s\(^{-2}\)) during the LDE [Owens et al., 1982, Figure 3].

With no time-independent model parameters and no problems with the reference state \(u = 0\), the differential travel time data were passed through the Kalman filter once to produce range-dependent current velocity and associated error fields. No significant range-dependent variation was found; that is, no variations in the estimated field were larger than the estimated errors: 5 cm s\(^{-1}\) rms at 300 m and 3 cm s\(^{-1}\) at depth. Resolution of the range-dependent terms was small.

The range-averaged current velocity profiles as a function of time are shown in Figure 23. The velocity above the main thermocline remains approximately ~20 cm s\(^{-1}\) (from North to West) for most of the experiment, but the velocity at depth changes from ~1 cm s\(^{-1}\) on day 218 to ~5 cm s\(^{-1}\) on day 234. Errors are approximately 2 cm s\(^{-1}\) above the thermocline and 1 cm s\(^{-1}\) at depth. The error reaches a minimum of 0.4 cm s\(^{-1}\) at the depth of the first baroclinic mode zero crossing (1250 m).

The three range-averaged mode amplitudes are plotted against time in Figure 24. The second baroclinic mode
amplitude does not differ significantly from zero, although it is responsible for the small perturbations in the profiles near the surface. The first baroclinic mode amplitude decreases by a factor of 2 over the course of the experiment. This variation is associated with the change in separation of the differential travel time data between the shallow turning rays and the deep turning rays (Figure 8). The error in the first baroclinic mode amplitude is large because it is determined primarily by the rather uncertain travel times associated with the shallow turning rays (Figure 9). The barotropic mode amplitude increases by a factor of 3, following the general trend in all the differential travel time data. The error for the range-averaged barotropic mode, ±0.4 cm s⁻¹, is smaller than the error on the first baroclinic mode amplitude because data from all the rays contribute.

The acoustically determined mode amplitudes can be compared with the baroclinic mode amplitudes from the deployment XBT (all three legs, days 212–216) and the AXBT survey (day 250). The latter mode amplitudes were calculated by simply fitting the geostrophic velocity shears calculated at 1000 m (XBT) and 650 m (AXBT) to the first baroclinic mode $Z^{(2)}$. The agreement at the beginning of the experiment is good; both the XBT and acoustics give first baroclinic mode amplitudes of 0.05 m s⁻¹. The XBT result is uncertain because of the assumption that linear gradients represent the XBT deployment field (Figure 11). The mode amplitudes for days 239 (acoustic) and day 250 (AXBT) are 0.03 and 0.07 m s⁻¹, respectively. The discrepancy is not significant because of the large errors on the acoustic results beyond day 235 (due to extrapolating the shallow ray data) and, more important, the 11-day time difference.

Depth and range-averaged velocity mode kinetic energies from the acoustic measurements and the XBT fields are compared in Table 4 with the a priori energies and LDE [Owens, 1985] and POLYMODE [Pinardi and Robinson, 1987] results. The LDE experimental area was relatively close geographically to the acoustic transmission path, but the POLYMODE area was 350 km to the south, farther from the active Gulf Stream recirculation region. This is reflected in the lower energies of the POLYMODE results relative to the acoustic and LDE results. The acoustic results compare favorably with the LDE total mean kinetic energy. Pinardi and Robinson found that barotropic and first baroclinic mode kinetic energies could change an order of magnitude in 10 days, so the changes by factors of 6 and 2 measured in this experiment between days 222 and 234 are not unexpected.

7. Discussion and Conclusions

The high-frequency variance analysis showed that the ray paths in opposite directions at 300 km range and for the conditions of this experiment are, for practical purposes, reciprocal. Internal wave and large-scale current shears did not significantly affect the ray paths.

Measurement errors have been reduced to acceptable levels in this experiment, but further improvement is possible. The major component of the low-frequency travel time error is due to high-frequency internal wave related fluctuations and can be reduced by more frequent sampling. For differential travel times, clock and interpolation errors are not insignificant. If the low frequency sum and difference travel time errors are reduced to 1 and 0.1 ms², respectively, simulations indicate that the error variances of the range-averaged and range-dependent sound speed field and the range-dependent current velocity fields will decrease only 20%. The errors on the range-averaged velocity profiles decrease to ±1 cm s⁻¹ above the main thermocline. The barotropic velocity mode uncertainty drops 30% while the error on the first baroclinic mode significantly decreases because the shallow rays, which largely determine the baroclinic mode amplitude, initially have larger errors. In general, reducing the measurement errors for this particular geometry will not significantly affect the estimated field uncertainties. Most of the uncertainty is due to lack of spatial resolution. For further improvement we require additional rays sampling different regions of the ocean as could be obtained by adding hydrophones vertically along each mooring. Hydrophones spaced 500 m apart would result in an order of magnitude more one-way travel time data. This would significantly improve the determination of the range-dependent sound

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<th>Table 4. Depth-Averaged Mode Kinetic Energy</th>
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<tr>
<td>$u_z$ along North leg</td>
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<td>(from XBTs, days 212–216)</td>
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<td>$u_z$ along North leg</td>
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<td>(from AXBTs, day 250)</td>
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<td>Acoustics</td>
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<td>A priori energies</td>
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<tr>
<td>LDE, Owens [1985]</td>
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<tr>
<td>31°N, 69°30'W</td>
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<tr>
<td>Mean total</td>
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<td>POLYMODE, Pinardi and Robinson [1987]</td>
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<td>29°N, 70°W</td>
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<td>Min</td>
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<td>Values are in cm² s⁻².</td>
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speed field. The current velocity fields would also be improved, but to an unknown extent.

The adoption of the Kalman filter provided a logical way of separating time-independent and time-dependent model parameters and a means of combining different types of data taken at different times to form optimal estimates of the model parameters and the fields of interest. For the sound speed case the Kalman filter showed that the two independent data sets, the deployment XBT and the acoustics, were consistent with each other and also with the ocean model. The Kalman filter also reduced the error in the estimated fields.

The inversions of differential travel times for current velocity were internally consistent; the acoustic data fit the ocean model within estimated errors. The acoustically measured baroclinic velocities agree with the XBT and AXBT baroclinic velocities to within the large uncertainties of the latter. Mode kinetic energies were consistent with previously measured values.

**APPENDIX A: SIGNAL CODING**

The transmitted signal was chosen to consist of periodic repetitions of a pseudo-random, phase-coded, linear maximal shift register sequence with the following characteristics: carrier frequency \( f_0 = 400 \text{ Hz} \) (exactly); digit length = four cycles of 400 Hz = 0.010 s; sequence length = 511 digits = 5.11 s; transmission length = 24 sequence periods = 122.64 s; and modulation phase angle \( \theta_0 = 87.408^\circ \). The transmitted signal corresponding to a single digit in the linear maximal sequence was defined as \( \sin(2\pi f_0 t + \theta_0) \) where \( t = 0 \) is the beginning of a digit and a positive (negative) phase shift corresponds to a logical "1" ("0") in the linear maximal sequence. This choice of modulation angle optimizes the signal-to-noise ratio and at the same time gives a smooth \( \sin^2 \) envelope to the line (power) spectrum of the periodic code.

This code can be processed to yield an output waveform that has no side lobes, i.e., no self-clutter [Spindel, 1979]. The received signal was complex demodulated and sampled at a two sample per digit rate (200 Hz) in the receivers. Twenty-two of the 24 sequence periods were coherently averaged; the initial and final periods were omitted from the average to avoid end effects. Two interleaved sequences consisting of one sample per digit were then constructed, separately correlated with a replica of the transmitted pseudo-random sequence, and finally recombined to give the processed arrival pattern.

The signal processing for each hydrophone was performed in situ and only the 5 (complex) data nearest the 32 largest peaks were saved on magnetic tape. This approach substantially reduced the tape storage relative to that required to store unprocessed data in the receivers. Additional information on the signal processing can be found in the work by Worcester et al. [1985].

**APPENDIX B: TRAVEL TIME AND ARRIVAL ANGLE TIME SERIES**

Time delay beam forming of the complex demodulated data from the hydrophone arrays was performed to obtain arrival angle information. The data, \( d_i(t) \), \( i = 1 \) to 4 hydrophones, for each of the 32 stored peaks, were interpolated using cubic splines to produce a continuous functional form for \( d_i(t) \). New beam-formed data at specified travel times \( t \) and beam angles \( \theta \) were calculated on a fine grid (\( \Delta t = 1.25 \text{ ms}, \Delta \theta = 2^\circ \)) according to

\[
d(t, \theta) = \frac{1}{4} \sum_{i=1}^{4} d_i(t + \delta t_i) \exp(-j2\pi f_0 \delta t_i)
\]

where \( \delta t_i = \Delta z_i \sin \theta / C_0 \) is the distance of the \( i \)th hydrophone from the array center and \( C_0 \) is the local speed of sound. The hydrophone spacing was 5.8 m \( \sim \frac{1}{2} \lambda \) where \( \lambda = C_0 \omega_0 = 3.75 \text{ m} \) is the acoustic wavelength. The central lobe of the beam pattern has its first zeros at \( \pm 9^\circ \); the angular precision is estimated to be 0.5° [Worcester et al., 1985]. The complex exponential term accounts for the complex demodulation of the signal. These formulas assume acoustic wave fronts are locally plane in the vicinity of the receiver. The \( d(t, \theta) \) was calculated only in \( t, \theta \) windows surrounding the expected arrival times and where data existed in time. These windows were allowed to move together on the \( t, \theta \) plane from reception to reception, largely in response to mooring motions. Local maxima in \( |d(t, \theta)| \) were found, and complex parabolas were fit to the adjacent points in \( t \) and \( \theta \) to determine peak parameters, \( t_p, \theta_p, \sigma_t, \sigma_\theta, \) and \( A_p \). \( A_p \) is the peak amplitude and the \( \sigma \) are the respective peak widths defined as \( \sqrt{R_p} \) A_p, where \( R_p \) is the radius of curvature of the peak in each respective direction. Signal-to-noise ratios, calculated using \( A_p \) and noise levels measured just prior to each reception, were increased by 6 dB to account for the coherent averaging of the beam forming [Worcester et al., 1985]. Peaks closer than 12.5 ms or 10° were dropped because of possible contamination due to interference. Peaks with \( \text{SNR} < 6 \text{ dB}, \sigma_t < 2.5 \text{ ms}, \sigma_\theta > 12.5 \text{ ms}, \sigma_\theta < 3^\circ, \text{ or } \sigma_\theta > 8^\circ \) were also dropped (nominal peak widths were 5 ms and 5°).

The remaining peaks were then tracked in time using a pattern recognition program based on a template made up of windows in \( t, \theta \). Initial window centers were specified by comparing the predicted \( t, \theta \) and the measured \( t, \theta \) in the first few receptions. Full window widths were 40 ms and 4°, enough to account for normal fluctuations in \( t, \theta \) from one reception to the next. The whole template was permitted to shift up to 50 ms and 1° between the bi-hourly receptions; window centers were allowed to move 20 ms and 0.5° within the template. If more than one peak was found in a window, neither was used. An exception to this was in tracking the axial arrival which was taken to be the largest last arrival with \( \theta \sim 0^\circ \). Twenty-four paths were tracked, although only 13 were finally used in the inversions; the others were too sparse (mostly surface-reflected, bottom-reflected). The beam forming more than doubled the total amount of useful data by separating arrivals close together in travel time, primarily the even numbered rays \( \pm 8, \pm 12, \) and \( \pm 14 \), which otherwise could not have been used. Typical tracked paths were 70–80% complete, while the best was 90%.

The raw travel times must be corrected for mooring motion and clock error. Figure B1 displays the raw travel times for the −11 path (West to North), the corrections
Fig. B1. Correcting the -11 path (west to north) one-way travel times. (a) The raw tracked peak series. (b) Corrections for the source motion $S$ (west), receiver motion $R$ (north), and the net motion of the two. The right-hand vertical axis shows the equivalent displacement along the ray path. (c) The source and receiver clock corrections. (d) The final corrected travel time series. This is equal to the raw travel time minus the corrections in figures B1b and B1c and a constant time correction which includes the nominal propagation delay and source and receiver electronic delays. Note that the vertical scales of panels Figures B1c and B1d are expanded by a factor of 4 relative to panels Figures B1a and B1b.

due to mooring motion and clock error, and the final corrected travel times. The mooring-motion correction refers the measured travel time to a fixed reference point using the plane wave approximation to remove the effect on travel time of relative source-receiver motion [Cornuelle, 1985]. This assumes that local ray curvature is small and acoustic wave fronts are plane and perpendicular to the ray path. This correction is calculated using the position of the acoustic interrogators as a function of time, the nominal vertical separations of instruments on the moorings, the tilt of the moorings, the ray path headings at each mooring, the ray angles in the vertical, the local sound speeds, and the nominal propagation time. In using the mooring tilt to better define instrument position, the mooring is assumed linear in the instrument section. Most of the change in raw travel time is due to the source and receiver moving toward each other by a maximum of 1100 m. The mooring motion correction taking into account these large-scale displacements is very important for one-way travel times. Although this procedure corrects for mooring displacements during the transmission-reception interval, i.e., the mooring velocity, the correction is very small: 0.1 ms in differential travel time or 1 mm s$^{-1}$ rms in current velocity (primarily at the $M_2$ tidal frequency), much smaller than other sources of errors.

The clock corrections over the time span plotted are at most 70 ms. The correction for each clock was determined by integrating over time the measured frequency difference between the low-power crystal oscillator and the rubidium standard. The instrument clocks were checked against Coordinated Universal Time (UTC) before and after the experiment; the maximum difference between the measured offset at recovery and the integrated correction was 1.8 ms. Over the 20 days of the reciprocal measurements, this error is estimated to be 0.3—0.6 ms. An additional timing correction, a constant, must be applied to account for electronic delays in the source and receiver.

The trend in the corrected data is the reverse of that in the uncorrected data. There is an increase of 0.1 s in travel time over the first 20 days. This corresponds to a $1 \text{m s}^{-1}$ decrease in sound speed, or a decrease of 0.2°C, averaged along the ray path. The high-frequency fluctuations are attributable primarily to internal waves and residual mooring motion noise.

**APPENDIX C: QUASI-GEOSTROPHIC VERTICAL STRUCTURE OF THE OCEAN MODEL**

The ocean model is based on the linearized quasi-geostrophic vorticity equation on a beta plane:

$$\frac{d}{dt} \left[ \nabla^2 \psi + \frac{\partial}{\partial z} \left( \frac{f \partial}{N^2(z)} \frac{\partial \psi}{\partial z} \right) \right] + \beta \frac{\partial \psi}{\partial x} = 0$$  \hspace{1cm} (C1)

with boundary conditions

Free surface \hspace{1cm} $w = -\frac{1}{g} \frac{\partial \psi}{\partial t}$ \hspace{1cm} $z = 0$

Flat bottom \hspace{1cm} $w = 0$ \hspace{1cm} $z = -D$

The stream function $\psi(x,t)$ is equal to $p'(x,t)/p_s(z)$, where $p'$ is the quasi-geostrophic pressure perturbation relative to a background hydrostatic state $\partial p/\partial z = -\rho_s g$. $N(z)$ is the buoyancy frequency. The first term in the equation is the relative vorticity, the second the vortex stretching, and the third the planetary vorticity. The vertical velocity in this perturbation framework is nonzero due to vortex stretching; assuming adiabatic flow

$$w = -\left( \frac{d \rho_s}{dz} \right)^{-1} \frac{\partial p'}{\partial t}$$

$$= -\frac{1}{N^2(z)} \frac{\partial^2 \psi}{\partial t \partial z}$$

where $p'$ is perturbation density. At the free surface this must equal the time rate of change of the vertical displacement of the interface. At the flat bottom the condition is zero normal velocity.
This equation for the most part appears applicable to mid-ocean, mid-latitude mesoscale processes. The assumed absence of (surface) forcing is a serious problem which is addressed in the main body of the text. The exclusion of the nonlinear terms naturally limits its predictive abilities, but with data in only two spatial dimensions the prediction capabilities cannot be used.

Equation (C1) is separable. Let

\[ \psi(x,t) = \phi(x,y,t)Z(z) \]

and use \( \lambda \) for the separation constant. Separating variables gives

\[ \frac{\partial}{\partial t} \left( \nabla^2 \phi - f_0 \lambda \phi \right) + \beta \frac{\partial \phi}{\partial x} = 0 \]

\[ \frac{d}{dz} \left( \frac{1}{N^2(z)} \frac{dZ(z)}{dz} \right) + \lambda Z(z) = 0 \]

The boundary conditions become

\[ \frac{dZ(z)}{dz} + \frac{N^2(z)}{g} Z(z) = 0 \quad z = 0 \]

\[ \frac{dZ(z)}{dz} = 0 \quad z = -D \]

The last three equations form a Sturm-Liouville problem with normal mode solutions \( Z_j(z) \) and eigenvalues \( \lambda_j \) (the linear Rossby wave modes, with radii of deformation \( \lambda_j \sqrt{f_0 u^2} \). The \( Z_j(z) \) form an orthogonal set of basis functions for the vertical structure of \( \psi(\cdot, -p') \). They are orthonormalized in a depth-averaged sense by setting

\[ \frac{1}{D} \int_0^D Z_j(z) Z_j(z) dz = \delta_{ij} \]

The stream function is now

\[ \psi(x,t) = \sum_j \phi_j(x,y,t)Z_j(z) \]

and the velocity in the +x direction is

\[ u(x,t) = -\frac{1}{f_0} \psi_x = -\frac{1}{f_0} \sum_j \phi_j(x,y,t)Z_j'(z) \]

where \( Z_j'(z) = Z_j(z) \) for emphasis. As discussed in the text, \( \phi_j \) is represented by a Fourier series with time-dependent amplitudes.

Modes for vertical displacement \( \xi \) are defined by

\[ Z_j^\xi(z) = \frac{1}{N^2(z)} \frac{dZ_j(z)}{dz} \]

Sound speed perturbation modes are then computed from the displacement of the background potential sound speed gradient,

\[ Z_j^\xi(z) = \frac{dC_{pot}(z)}{dz} Z_j^\xi(z) \]

where \( C_{pot}(z) \) is the potential sound speed.

**APPENDIX D: TIME-INDEPENDENT INVERSE PROCEDURE**

Let \( \tilde{\beta} = [d, \tilde{d}, \tilde{m}] \) represent the quantities being minimized. The least squares procedure expects all components of \( \tilde{\beta} \) to be uncorrelated Gaussian random variables with zero mean and unit variance. Some \( \tilde{\beta}_i \) will most likely have larger expected uncertainties than others: Some data are less accurate than others, and some model parameters may be expected to vary more widely than others about the expected value of zero. Changes in dimensional units produce the same effect. Weighting, including nondimensionalization, is necessary so all the \( \tilde{\beta}_i \) are uncorrelated and have expected variances of unity.

We rewrite the problem as

\[ \tilde{d} = \tilde{G}\tilde{m} + \tilde{e} \]

The tilde indicates weighted quantities defined as

\[ \tilde{d} = E^{-\frac{1}{2}}d \quad \tilde{e} = E^{-\frac{1}{2}}e \]

\[ \tilde{G} = E^{-\frac{1}{2}}GW^{-\frac{1}{2}} \]

and

\[ \tilde{m} = W^{-\frac{1}{2}}\tilde{m} \]

\( E \) is the covariance of the data error \( (ee') \), here assumed diagonal. Weighting by the inverse data error is done so that the more accurate data count more than the inaccurate data. \( W \) is an a priori estimate of the expected covariance of the model parameters \( (mm') \) also assumed diagonal. The weighted covariance matrices \( (\tilde{e}\tilde{e}') \) and \( (\tilde{m}\tilde{m}') \) equal the identity so that when the weighted norms, \( \|d-\tilde{d}\|^2 \) and \( \|\tilde{m}\|^2 \), are minimized, values greater than unity are heavily penalized while values less than unity are not.

A unique inverse of \( \tilde{G} \) is obtained by performing a singular value decomposition (SVD) and tapering reciprocal eigenvalues. This inverse is termed the generalized inverse and is denoted by \( \tilde{G}^{-x} \):

\[ \tilde{G}^{-x} = UAV^T \]

\[ \Lambda^{-x} = \frac{\lambda_i}{\lambda_i^2 + 1} \]

where the \( \lambda_i \) are the eigenvalues of \( \tilde{G} \). Estimates, denoted by a hat (absence of a hat denotes the true value) of the model parameters can be calculated using

\[ \hat{m} = W^{\frac{1}{2}}\tilde{m} \]

where

\[ \hat{m} = \tilde{G}^{-x}\tilde{d} = \tilde{G}^{-x}(G\hat{m} + \tilde{e}) = \tilde{R}\hat{m} + \tilde{G}^{-x}\tilde{e} \]

\( \tilde{R} \) is called the (weighted) resolution matrix. It shows that neglecting errors, the estimated model parameters are a weighted average of the true model parameters.

The estimated uncertainty in the estimated model parameters, weighted and unweighted, are
\[ \hat{H} = (\mathbf{m} - \hat{\mathbf{m}})^T (\mathbf{m} - \hat{\mathbf{m}}) \]
\[ = (1 - \mathbf{R}) \mathbf{m} \mathbf{m}^T (1 - \mathbf{R}) + \mathbf{G}^{-1} (\varepsilon \varepsilon^T) \mathbf{G}^{-1} = (1 - \mathbf{R}) \]
\[ \hat{H} = (\mathbf{m} - \hat{\mathbf{m}})(\mathbf{m} - \hat{\mathbf{m}})^T \]
\[ = \mathbf{W}^{1/2} \hat{\mathbf{H}} \mathbf{W}^{1/2} \]

(Recall that the weighted data and model error covariance matrices are equal to the identity.)

Estimates of the desired fields and of the mean square errors in these estimates can now be calculated, for instance, for velocity,
\[ \tilde{u}(x,z,t) = \sum_j \tilde{m}_j^u(t) P^u_j(x,z) \]
\[ \langle (u - \tilde{u})^2 \rangle = \sum_i \sum_j P^u_i \tilde{m}_j^u \tilde{m}_j^u \]

Finally, weighted data residuals
\[ r_i = (d_i - \tilde{d}_i)/\epsilon_i \]
are calculated using \( \tilde{d} = \mathbf{G} \hat{\mathbf{m}} \). These are a direct measure of the misfit between the data and the model. Weighted data residuals have been assumed to be normally distributed with zero mean and unit standard deviation. The \(|r_i|\) is considered suspicious if it is larger than \( \sim 1.2 \). If it is larger than \( \sim 1.8 \), the model is assumed to be lacking sufficient degrees of freedom to account for all the information in the data; the data are inconsistent with the model, implying that the model is inadequate (these threshold values will naturally depend on the number of data). Large residuals in preliminary sound speed inversions were the motivation for adding the triangle basis functions.

**APPENDIX E: TIME-DEPENDENT INVERSE PROCEDURE**

Let expected values of \( \mathbf{m}(t) \) be close to some \( \mathbf{c}(t) \) at time \( t \). Furthermore, let \( \mathbf{C}(t) \) be an estimate of the uncertainty of \( \mathbf{c}(t) \) as an estimate of \( \mathbf{m}(t) \). Then the weighted perturbation model parameters
\[ \hat{\mathbf{m}}(t) = C^{-1/2} \left[ \mathbf{m}(t) - \mathbf{c}(t) \right] \]
will have zero expected value and a covariance matrix equal to the identity.

Rewrite the basic problem
\[ \tilde{d} = \mathbf{G} \hat{\mathbf{m}} + \varepsilon \]
where the weighted perturbation quantities are
\[ \tilde{d} = \mathbf{E}^{-1/2} (d - \mathbf{G} \mathbf{c}) \]
\[ \mathbf{G} = \mathbf{E}^{-1/2} \mathbf{G} C (\mathbf{G} C)^{1/2} \]
and
\[ \mathbf{m} = C^{1/2} \hat{\mathbf{m}} + \mathbf{c} \]
\[ \mathbf{E} \]

is the covariance of the data error \( (\varepsilon \varepsilon^T) \) here assumed diagonal. Using the generalized inverse,
\[ \hat{\mathbf{m}} = \mathbf{G}^{-1} \tilde{d} \]
\[ \hat{\mathbf{m}} = C^{1/2} \hat{\mathbf{m}} + \mathbf{c} \]
\[ C^{1/2} \]

is the upper triangular Cholesky factor of \( \mathbf{C} [\text{Lawson and Hanson, 1974}] \).

Exactly how are \( \mathbf{c} \) and \( \mathbf{C} \) specified? B. Cornuelle (personal communication, 1985) suggests that in the absence of data (and any dynamics), time-dependent model parameters \( \mathbf{m}(t) = C^{1/2}(t) \hat{\mathbf{m}}(t) + \mathbf{c}(t) \) should decay slowly with a time constant \( \tau \) to reference values, and the uncertainties \( \mathbf{C}(t) \) should increase slowly to a priori variances (modified persistence). This assumes the reference state is the climatological average and that the a priori variances represent the mesoscale fluctuations about the climatological reference state. The rules for specifying \( \mathbf{c} \) and \( \mathbf{C} \) are
\[ \mathbf{c}(t) = \begin{cases} 1 - \frac{t}{\tau} \hat{\mathbf{m}}(0) & t < \tau \\ 0 & t > \tau \end{cases} \]
and
\[ \mathbf{C}(t) = \begin{cases} \frac{t}{\tau} (\mathbf{W}_c - \mathbf{F}_0(0)) & t < \tau \\ \mathbf{W}_c - \mathbf{F}_0(0), \ i \neq j & t > \tau \end{cases} \]

The present data assimilation step occurs at time \( t \), while time 0 corresponds to the previous data assimilation step. The diagonal terms \( \mathbf{C}(t) \) revert to climatological variance estimates while the off-diagonal terms revert to zero. For time-dependent model parameters a time constant \( \tau = 10 \) days is used. This means that if there are no data for 10 days, the perturbation fields will revert to zero and the uncertainties or errors in the fields will revert to what is expected for mesoscale variability about the climatological reference state. Another way to state this is that there is a 10-day memory; after 10 days the field no longer contains information from the first day's data.

For time-independent model parameters the prediction rule is very simply persistence (\( \tau = \infty \)): \( \mathbf{c} = \hat{\mathbf{m}} \) and \( \mathbf{C} = \hat{\mathbf{H}} \). If new data provide new information, perhaps in the form of a different noise realization, the estimate of \( \hat{\mathbf{m}} \) is always improved, as reflected by the decreased uncertainty. If a long time series were available, the time constant of the deep modes should be set to what might be an appropriate deep ocean time scale, say, 1 year. This is one possible way of determining a long-term warming or cooling trend in the deep ocean.

An advantage of using the Kalman filter is that the a priori statistics of the model parameters relative to the reference state are not so important. Rather, what is important is the rate at which model parameters are expected to change between data assimilation steps: a more robust quantity. This rate may be determined by the modified persistence rule used here or by a dynamical ocean model (B. Cornuelle, personal communication, 1985). The statistics of interest now are the uncertainties of the model parameters relative to a time-dependent reference state. Knowledge of a true mean reference state is not so important, the data will produce an unbiased estimate of the true mean state.

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